

The Complexity of Recognizing Incomplete Single-Crossing Preferences

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Abstract

We study the complexity of deciding if a given profile of incomplete votes (i.e., a profile of partial orders over a given set of alternatives) can be extended to a single-crossing profile of complete votes (total orders). This problem models settings where we have partial knowledge regarding voters' preferences and we would like to understand whether the given preference profile may be single-crossing. We show that this problem admits a polynomial-time algorithm when the order of votes is fixed and the input profile consists of top orders, but becomes NP-complete if we are allowed to permute the votes and the input profile consists of weak orders or independent-pairs orders. Also, we identify a number of practical special cases of both problems that admit polynomial-time algorithms.

1 Introduction

An important job for a designer of a multi-agent system is identifying a good method of aggregating the agents' preferences. It is well-known that this is not an easy task, at least if agents' preferences can be arbitrary total orders over the available alternatives: every preference aggregation mechanism for this setting exhibits undesirable behavior on some inputs (Arrow 1951). However, the designer's task becomes much easier when agents' preferences possess additional structure.

For instance, the well-known class of *single-peaked preferences* (Black 1958) admits a voting rule that always selects a Condorcet winner (an alternative that is preferred to every other alternative by a majority of voters) and is strategyproof (Moulin 1991). Moreover, single-peaked preferences admit efficient algorithms for problems that are more complex than selecting a single winner and that are known to be hard for general preferences, such as choosing a good ranking of the alternatives (Brandt et al. 2010) or a representative committee (Betzler, Slinko, and Uhlmann 2013).

In this paper, we focus on another restricted preference domain, namely, that of *single-crossing preferences*. A preference profile is single-crossing with respect to a fixed ordering of voters if for every pair of alternatives (a, b) it holds

that all voters who prefer a to b precede all voters who prefer b to a or vice versa. A profile is single-crossing if the votes can be permuted so as to achieve the single-crossing property. Single-crossing preferences, originally introduced by Mirrlees (1971) and Roberts (1977), arise in situations where voters and candidates are spread over a spectrum of opinions—say, from extreme left-wing ones to extreme right-wing ones—and left-leaning voters prefer left-leaning candidates to right-leaning ones, and the other way round for the right-leaning voters. While this domain is perhaps not as well-known as that of single-peaked preferences, it has many of the same desirable properties: for instance, under single-crossing preferences the majority relation is transitive (Mirrlees 1971), and single-crossing preferences admit efficient algorithms for several voting problems that are hard for the general domain (Cornaz, Galand, and Spanjaard 2013; Skowron et al. 2013; Magiera and Faliszewski 2014).

However, in practice we rarely have access to voters' full preferences: voters are far more likely to only report some part of their preference order, e.g., rank a few top alternatives or report a small number of pairwise comparisons. Indeed, in an overwhelming majority of data sets in `PreFLib` (Mattei and Walsh 2013) preference profiles contain partial orders. This phenomenon is recognized by computational social choice researchers, who showed that many of the positive results that are known to hold for complete preference profiles can be extended to partial preference profiles (Baumeister et al. 2012; Narodytska and Walsh 2014). It also motivated research on the *possible/necessary winner problem* (Konczak and Lang 2005; Betzler and Dorn 2010; Xia and Conitzer 2011; Baumeister and Rothe 2012), where we ask whether a given candidate wins in some/all extensions of a given profile of partial votes to a profile of full votes, under a particular voting rule. In a similar vein, we can ask if a profile of partial votes can be extended so that it enjoys a particular structural property, such as being single-peaked/single-crossing, and, if the answer is positive, whether we can identify an ordering of candidates/voters witnessing this. Answering this question would tell us whether voters' preferences may be essentially one-dimensional in nature; if this answer is positive, we may be able to make a reasonably good decision quickly and without eliciting full preferences.

For incomplete single-peaked preferences, the complex-

ity of this problem has been investigated by Lackner (2014), who proved that it is NP-complete when the input may consist of arbitrary partial orders, and more recently by Fitzsimmons (2014), who showed it to be polynomial-time solvable for weak orders. The goal of our paper is to initiate the complexity-theoretic investigation of this problem for incomplete single-crossing preferences.

Our Contribution. We consider the complexity of deciding whether a given profile of partial orders can be extended to a profile of total orders that is single-crossing. We investigate this problem both for the setting where we are allowed to permute the votes so as to achieve the single-crossing property and for the setting where the desired ordering of the votes is fixed.

We first focus on the case where the ordering of the votes is provided as part of the input, and show that our problem admits an efficient algorithm when the input profile consists of top orders, or when no input vote contains an antichain of size 3 and for every pair of candidates there is at least one voter who is able to compare them. We then turn to the problem of checking whether a given profile of partial orders can be extended to a profile of total orders that is single-crossing with respect to *some* ordering of votes. We show that this problem is NP-complete, even if all votes in the input are weak orders, or if in each input vote all antichains of size 2 are pairwise disjoint. Given these hardness results, we focus on top orders, and obtain polynomial-time algorithms under mild additional assumptions on voters' preferences. We show that, given a profile of top orders \mathcal{R} , we can efficiently decide whether it can be extended to a single-crossing profile of total orders if \mathcal{R} contains at least one full vote and (i) we seek an ordering of the votes where some such vote appears first or (ii) the input profile is *narcissistic*, i.e., each candidate is ranked first by at least one voter.

We also investigate alternative extensions of the single-crossing property to the domain of partial votes. In particular, we define the notion of a *weakly single-crossing profile*, and show that such profiles can be detected efficiently.

Relevance of Our Study. We believe that understanding the single-crossing property in the context of partial preference orders is important in its own right. However, our research also has a more direct motivation: knowing that a profile of partial preference orders can be extended to a single-crossing one can simplify the winner determination process, both in single-winner and in multi-winner elections.

Consider a profile of top orders in a single-winner election. If we know that the votes can be extended to a single-crossing profile for a given voter order, then we can find the median vote in this order and pick its top candidate as the winner. This candidate is a possible Condorcet winner and, thus, a natural one to select.

For the case of multi-winner elections, Skowron et al. (2013) have shown an efficient winner determination algorithm for the voting rule of Chamberlin and Courant (1983), for the case of single-crossing elections (in the general setting, the rule is NP-hard (Procaccia, Rosenschein, and Zohar 2008; Lu and Boutilier 2011)). Their algorithm focuses on the top parts of the votes, but requires the

order witnessing that the election is single-crossing. Thus, if we could find an order witnessing that a profile of top orders can be extended to a single-crossing profile, then we could use the algorithm of Skowron et al. (2013).

There is a further added benefit of considering the single-crossing property in the context of partial preference orders. Intuitively, when voters cast partial preference orders, they only specify pairwise comparisons that they truly care about. Consequently, the resulting profiles are much more likely to satisfy various structural properties (such as being single-peaked/single-crossing) than profiles where voters are forced to rank candidates that they do not care about (and may therefore rank them in a way that hides the true preference structure).

Related Work. Both single-peaked and single-crossing preferences can be recognized in polynomial time if the input is a collection of total orders (Bartholdi and Trick 1986; Escoffier, Lang, and Öztürk 2008; Elkind, Faliszewski, and Slinko 2012; Bredereck, Chen, and Woeginger 2013b). The problem becomes much more difficult if we ask whether a given preference profile is close to being single-peaked or single-crossing, or, more generally, close to belonging to some restricted domain, for an appropriate notion of distance; indeed, many (though not all!) variants of this problem are known to be NP-hard (Erdélyi, Lackner, and Pfandler 2013; Bredereck, Chen, and Woeginger 2013a; Faliszewski, Hemaspaandra, and Hemaspaandra 2014). Both single-peaked and single-crossing preferences arise in societies that are, in some sense, one-dimensional; however, the two notions are distinct, in the sense that there are single-peaked elections that are not single-crossing and vice versa (Mirrlees 1971); see also the work of Elkind et al. (2014).

2 Preliminaries

For each integer k , we denote the set $\{1, \dots, k\}$ by $[k]$. Let C be a finite set of candidates (alternatives). A (*strict*) *partial order* is a binary relation \succ over C that has the following properties: for every $a, b, c \in C$ (i) $a \not\succeq a$; (ii) $a \succ b$ implies $b \not\succeq a$; (iii) $a \succ b$ and $b \succ c$ implies $a \succ c$. We say that a pair of alternatives a, b is *comparable* in \succ if $a \succ b$ or $b \succ a$; otherwise we say that a and b are *incomparable* in \succ and write $a \perp_{\succ} b$. A partial order \succ is said to be *total* if $a \succ b$ or $b \succ a$ for every $a, b \in C$. A total order \succ is an *extension* of a partial order \succ' if for every pair of alternatives a, b such that $a \succ' b$ it holds that $a \succ b$.

When $a \succ b$, we say that \succ ranks a above b . For readability, we will often denote a generic partial order by r and write $a \succ_r b$ or $r : a \succ b$ when r ranks a above b .

A partial order \succ is said to be a *weak order* if for all $a, b, c \in C$ it holds that $a \perp_{\succ} b$ and $b \perp_{\succ} c$ implies $a \perp_{\succ} c$. Equivalently, in a weak order \succ all candidates are partitioned into several equivalence classes C_1, \dots, C_k so that for $a \in C_i, b \in C_j$ we have $a \perp_{\succ} b$ if $i = j$ and $a \succ b$ if $i < j$. Weak orders can be understood as total orders with ties allowed. A *top order* is a weak order where $|C_1| = \dots = |C_{k-1}| = 1$. Intuitively, top orders correspond to a voter ranking some of her most preferred alternatives, and leaving the remaining alternatives unranked. Thus, we refer to the

candidates in $\bigcup_{i=1}^{k-1} C_i$ as the *ranked candidates*.

A set $C' \subseteq C$ is said to be an *antichain* in \succ if $a \perp_{\succ} b$ for all $a, b \in C'$. A partial order \succ is said to be an *independent pairs order* if all antichains of size 2 in \succ are pairwise disjoint (this implies that \succ has no antichains of size 3).

Single-Crossing Property. A list $\mathcal{R} = (r_1, \dots, r_n)$ of (partial) orders is called a *(partial) profile*. We refer to elements of $[n]$ as *voters*: the order r_i is the vote of voter i .

Definition 1. A profile $\mathcal{R} = (r_1, \dots, r_n)$ of total orders over a candidate set C is said to be *single-crossing* with respect to a total order \sqsubset on $[n]$ if for every pair of candidates $a, b \in C$ such that the first voter in \sqsubset prefers a to b it holds that the voters who prefer a to b precede in \sqsubset the voters who prefer b to a . A profile $\mathcal{R} = (r_1, \dots, r_n)$ of total orders is said to be *single-crossing* if there exists a total order \sqsubset on $[n]$ such that \mathcal{R} is single-crossing with respect to \sqsubset .

Definition 2. A profile $\mathcal{R} = (r_1, \dots, r_n)$ of partial orders over a candidate set C is said to be *single-crossing* with respect to a total order \sqsubset on $[n]$ if there exists a profile $\widehat{\mathcal{R}} = (\widehat{r}_1, \dots, \widehat{r}_n)$ of total orders, where \widehat{r}_i is an extension of r_i for each $i \in [n]$, that is single-crossing with respect to \sqsubset . \mathcal{R} is said to be *single-crossing* if there exists a total order \sqsubset on $[n]$ such that \mathcal{R} is single-crossing with respect to \sqsubset .

Computational Problems. The goal of this paper is to study the computational complexity of the following two problems (and their special cases):

PARTIAL ORDER SINGLE-CROSSING CONSISTENCY
— FIXED ORDER (PO-SCC-F):

Given a candidate set C , a profile $\mathcal{R} = (r_1, \dots, r_n)$ of partial orders over C , and a total order \sqsubset on $[n]$, decide whether \mathcal{R} is single-crossing with respect to \sqsubset .

PARTIAL ORDER SINGLE-CROSSING CONSISTENCY
(PO-SCC):

Given a candidate set C and a profile $\mathcal{R} = (r_1, \dots, r_n)$ of partial orders over C , decide whether \mathcal{R} is single-crossing.

We are also interested in special cases of PO-SCC-F and PO-SCC where the input profile contains: (i) weak orders only (WO-SCC-F/WO-SCC), (ii) top orders only (TO-SCC-F/TO-SCC), (iii) independent-pairs orders only (IP-SCC-F/IP-SCC).

We omit some proofs due to space constraints. The omitted proofs appear in the full version of the paper.

3 Fixed Order of Votes

Before we move to the computational results, let us illustrate how counterintuitively partial orders can behave with respect to the single-crossing property. Let us define a relaxed variant of this property, tailored to partial orders.

Definition 3. A profile $\mathcal{R} = (r_1, \dots, r_n)$ of partial orders over a candidate set C is *seemingly single-crossing* with respect to a total order \sqsubset over $[n]$ if for every pair of candidates $a, b \in C$ the voters can be divided into two (possibly empty) consecutive intervals with respect to \sqsubset so that (i) in one of these intervals each voter either prefers a to b or indicates that a and b are incomparable, and (ii) in the other

interval each voter either prefers b to a or indicates that a and b are incomparable. A profile $\mathcal{R} = (r_1, \dots, r_n)$ of partial orders is *seemingly single-crossing* if it is *seemingly single-crossing with respect to some total order \sqsubset over $[n]$* .

A profile of total orders is single-crossing if and only if it is *seemingly single-crossing*. One might expect that the same is true for profiles of partial orders, i.e., that a profile of partial orders that is *seemingly single-crossing with respect to some order of votes \sqsubset* can be extended to a profile of total orders that is single-crossing with respect to \sqsubset . However, the next example shows that this is not the case.

Example 1. Let $C = \{a, b, c\}$ and consider the following profile $\mathcal{R} = (r_1, r_2, r_3, r_4)$ of partial orders:

$$r_1: a \succ b \succ c, \quad r_2: c \succ b, \quad r_3: b \succ a, \quad r_4: a \succ c.$$

It is easy to see that \mathcal{R} is *seemingly single-crossing* with respect to the order $1 \sqsubset 2 \sqsubset 3 \sqsubset 4$. However, \mathcal{R} cannot be extended to a profile of total orders $(\widehat{r}_1, \widehat{r}_2, \widehat{r}_3, \widehat{r}_4)$ that is single-crossing with respect to \sqsubset . Indeed, $a \succ_{r_1} b, b \succ_{r_3} a$ implies that \widehat{r}_4 would have to rank b above a , and $b \succ_{r_1} c, c \succ_{r_2} b$ means that \widehat{r}_4 would have to rank c above b . By transitivity, it follows that \widehat{r}_4 ranks c above a , but this is impossible, since $a \succ_{r_4} c$.

This argument does not show that \mathcal{R} is not single-crossing. In fact, \mathcal{R} is single-crossing with respect to a different order of voters, namely $1 \sqsubset' 2 \sqsubset' 4 \sqsubset' 3$, as witnessed by the following profile $(\widehat{r}_1, \widehat{r}_2, \widehat{r}_3, \widehat{r}_4)$ of total orders (for convenience, the votes below are listed according to \sqsubset'):

$$\begin{aligned} \widehat{r}_1: a \succ b \succ c, & & \widehat{r}_2: a \succ c \succ b, \\ \widehat{r}_4: a \succ c \succ b, & & \widehat{r}_3: c \succ b \succ a. \end{aligned}$$

However, we can modify \mathcal{R} so that it remains *seemingly single-crossing*, but is not single-crossing with respect to *any* order of voters. Specifically, set $C' = \{a, b, c, d, e, f\}$, and consider the following profile $\mathcal{R}' = (r'_1, r'_2, r'_3, r'_4)$ of partial orders over C' , which is obtained by prepending a single-crossing profile of total orders over $\{d, e, f\}$ to \mathcal{R} :

$$\begin{aligned} r'_1: d \succ e \succ f \succ a \succ b \succ c, \\ r'_2: e \succ d \succ f, \quad f \succ a, \quad f \succ c \succ b, \\ r'_3: e \succ f \succ d, \quad d \succ c, \quad d \succ b \succ a, \\ r'_4: f \succ e \succ d, \quad d \succ b, \quad d \succ a \succ c. \end{aligned}$$

It is easy to see that \mathcal{R}' is *seemingly single-crossing* with respect to $1 \sqsubset 2 \sqsubset 3 \sqsubset 4$. Further, the $\{d, e, f\}$ -parts of orders in \mathcal{R}' ensure that the only orders for which \mathcal{R}' is *seemingly single-crossing* are $1 \sqsubset 2 \sqsubset 3 \sqsubset 4$ and $4 \sqsubset 3 \sqsubset 2 \sqsubset 1$. Thus, no extension of \mathcal{R} is single-crossing.

Example 1 shows that, to solve PO-SCC-F, it is not sufficient to check whether the input profile is *seemingly single-crossing*. Indeed, we have been unable to determine the complexity of PO-SCC-F for unrestricted inputs. However, we can show that this problem becomes polynomial-time solvable if we additionally assume that no order in the input profile contains an antichain of size 3, and no pair of candidates is incomparable in every vote.

Theorem 2. *One can determine whether an n -voter m -candidate profile of partial orders \mathcal{R} is single-crossing with respect to a given order \sqsubset on $[n]$ in $\mathcal{O}(m^2 \cdot n \cdot (m+n))$ time under the following two conditions: (1) \mathcal{R} does not contain a vote with an antichain of size 3, and (2) no pair of candidates is incomparable in every vote.*

If a profile does not satisfy conditions (1) and (2) in the statement of Theorem 2, but there are only few antichains of size 3 and “fully” incomparable candidate pairs, we can still solve this problem efficiently. Let a denote the total number of antichains of size 3 in \mathcal{R} and let b denote the number of candidate pairs that are incomparable in all votes.

Corollary 3. *The PO-SCC-F problem can be solved in $\mathcal{O}(2^{a+b} \cdot m^2 \cdot n \cdot (m+n))$ time.*

For top orders, a stronger result is true: TO-SCC-F is polynomial-time solvable with no additional constraints on the input profile. Moreover, for top orders the phenomenon illustrated in Example 1 does not arise: every seemingly single-crossing profile of top orders is single-crossing. To prove this, we will now present an algorithm that, given a profile \mathcal{R} of top orders that is seemingly single-crossing with respect to an ordering \sqsubset , explicitly constructs an extension of \mathcal{R} that is single-crossing with respect to \sqsubset . We first describe a subroutine \mathcal{E} used by our algorithm.

Algorithm \mathcal{E} : The algorithm takes as input a profile $\mathcal{R} = (r_1, \dots, r_n)$ of top orders, where r_1 is a total order, and an order \sqsubset over $[n]$ such that $1 \sqsubset i$ for each $i \in \{2, \dots, n\}$. It computes a profile of total orders as follows:

1. It orders the votes in \mathcal{R} according to \sqsubset to obtain a profile $\mathcal{S} = (s_1, \dots, s_n)$; note that $r_1 = s_1$.
2. It sets $\widehat{s}_1 = s_1$ and for each $i \in \{2, \dots, n\}$ (in the ascending order), it extends s_i to \widehat{s}_i by ranking all the unranked candidates as in \widehat{s}_{i-1} (note that by the time it processes s_i , \widehat{s}_{i-1} is a total order).
3. It returns $(\widehat{s}_1, \dots, \widehat{s}_n)$.

Theorem 4. *There is a polynomial-time algorithm that given a profile \mathcal{R} of top orders that is seemingly single-crossing with respect to an order \sqsubset on $[n]$, outputs an extension of \mathcal{R} that is single-crossing.*

Proof. Let C be a set of candidates and let $\mathcal{R} = (r_1, \dots, r_n)$ be a profile of top orders over C that is seemingly single-crossing with respect to an order \sqsubset on $[n]$. Without loss of generality, we assume that \sqsubset is given by $1 \sqsubset 2 \sqsubset \dots \sqsubset n$. To find a single-crossing extension $\widehat{\mathcal{R}} = (\widehat{r}_1, \dots, \widehat{r}_n)$ of \mathcal{R} , we first compute an extension \widehat{r}_1 of r_1 :

1. Set $\widehat{r}_1 = r_1$.
2. For each $i = 2, \dots, n$, if r_i ranks some candidates that \widehat{r}_1 does not yet rank, append these candidates to \widehat{r}_1 (in order of their appearance in r_i).
3. If \widehat{r}_1 still does not rank all the candidates, append them to \widehat{r}_1 in an arbitrary order.

Now we have a profile $\mathcal{R}' = (\widehat{r}_1, r_2, \dots, r_n)$ of top orders, where \widehat{r}_1 is a total order. We run Algorithm \mathcal{E} on \mathcal{R}' to obtain a profile of total orders. Note that in Algorithm \mathcal{E} we

set $\mathcal{S} = \mathcal{R}'$, and therefore this profile, which we will denote by $\widehat{\mathcal{R}} = (\widehat{r}_1, \widehat{r}_2, \dots, \widehat{r}_n)$, is an extension of \mathcal{R} . We claim that $\widehat{\mathcal{R}}$ is single-crossing with respect to \sqsubset .

Suppose that $\widehat{\mathcal{R}}$ is not single-crossing and let ℓ be the largest index such that $(\widehat{r}_1, \dots, \widehat{r}_{\ell-1})$ is single-crossing. Thus, $(\widehat{r}_1, \dots, \widehat{r}_\ell)$ is not single-crossing and there exists a pair a, b of candidates such that

$$\widehat{r}_1: a \succ b, \quad \widehat{r}_{\ell-1}: b \succ a, \quad \widehat{r}_\ell: a \succ b.$$

Candidates a and b are ranked differently in $\widehat{r}_{\ell-1}$ and \widehat{r}_ℓ , so Algorithm \mathcal{E} could not have derived the ranking $a \succ b$ in \widehat{r}_ℓ from $\widehat{r}_{\ell-1}$. Hence, in r_ℓ we also have $a \succ b$. Since \widehat{r}_1 and $\widehat{r}_{\ell-1}$ rank a and b differently and given how vote \widehat{r}_1 is computed, there must be a k , $1 \leq k < \ell - 1$ such that $r_k: a \succ b$ and neither a nor b are ranked in any r_i , $i \in [k-1]$. Consequently, the triple $(r_k, r_{\ell-1}, r_\ell)$ witnesses that \mathcal{R} is not seemingly single-crossing with respect to $1 \sqsubset 2 \sqsubset \dots \sqsubset n$, a contradiction with our assumption. Thus, the algorithm outputs a single-crossing extension of \mathcal{R} . \square

4 Arbitrary Order of Voters

We will now consider the scenario where the ordering of the votes is not given in the input, and we have to decide whether the given profile is single-crossing with respect to *some* ordering of the votes. Note that in this setting we can assume that all votes in the input profile \mathcal{R} are pairwise distinct, as we can simply remove all duplicates without changing the answer. Therefore, we can view \mathcal{R} as a *set* of votes, and identify a voter i with her vote r_i . In particular, it will sometimes be convenient to write $r_i \sqsubset r_j$ in place of $i \sqsubset j$.

Hardness Results The problem PO-SCC turns out to be NP-complete. To show this, we will provide a reduction from the BETWEENNESS problem, defined below, which is known to be NP-complete (Opatrný 1979).

BETWEENNESS:

Given a set $S = \{s_1, \dots, s_m\}$ and a set T of triples over S , decide whether there exists a total order $<$ over S such that for each triple (s_i, s_j, s_k) in T it holds that either $s_i < s_j < s_k$ or $s_k < s_j < s_i$.

To reduce BETWEENNESS to PO-SCC, we use instances of the following gadget. Let $\mathcal{R} = (r_1, r_2, r_3)$ be a profile over $C = \{a, b, c\}$ such that: $r_1: a \succ b \succ c$, $r_2: b \succ a \succ c$, and $r_3: c \succ b \succ a$. It is easy to verify that \mathcal{R} is single-crossing with respect to exactly two orders: $1 \sqsubset 2 \sqsubset 3$ and its reverse.

Theorem 5. *PO-SCC is NP-complete.*

Proof. Clearly, this problem is in NP. To show that it is NP-hard, we provide a reduction from BETWEENNESS.

Let $I = (S, T)$ be an instance of BETWEENNESS, where $S = (s_1, \dots, s_m)$ and $T = (t_1, \dots, t_n)$ is a set of triples over S . The idea of our proof is to form a profile where the voters correspond to the elements of the set S and the constraints from the set T are implemented within the partial orders using the gadget described just before the theorem statement. We let $D = A \cup B \cup C$, where $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$, and $C = \{c_1, \dots, c_n\}$, and form a partial profile $\mathcal{R} = (r_1, \dots, r_m)$ over D as follows:

1. For each $\ell \in [m]$, each $i, j \in [n]$, $i < j$, each $x \in \{a_i, b_i, c_i\}$ and each $y \in \{a_j, b_j, c_j\}$, we set $r_\ell : x \succ y$.
2. For each triple $t_\ell = (s_i, s_j, s_k) \in T$, we set:

$$r_i : a_\ell \succ b_\ell \succ c_\ell, \quad r_j : b_\ell \succ a_\ell \succ c_\ell, \quad r_k : c_\ell \succ b_\ell \succ a_\ell.$$
3. For each $\ell \in [m]$ every pair of candidates not mentioned in the previous two points are incomparable in r_ℓ .

We claim that \mathcal{R} is single-crossing if and only if I is a “yes”-instance of BETWEENNESS. First, assume that \mathcal{R} is single-crossing with respect to some order \sqsubset . By construction, for each triple $t_\ell = (s_i, s_j, s_k) \in T$, we have either $r_i \sqsubset r_j \sqsubset r_k$ or $r_k \sqsubset r_j \sqsubset r_i$. This means that an order $<$ over S such that $s_x < s_y$ if and only if $r_x \sqsubset r_y$ witnesses that I is a “yes”-instance of the BETWEENNESS problem.

On the other hand, assume that I is a “yes”-instance of the BETWEENNESS problem and that some order $<$ over S witnesses this. We define an order \sqsubset over $\{r_1, \dots, r_m\}$ so that $r_x \sqsubset r_y$ if and only if $s_x < s_y$. To show that \mathcal{R} is single-crossing with respect to \sqsubset , we will now extend \mathcal{R} to a profile of total orders as follows. Consider a triple $t_\ell = (s_i, s_j, s_k) \in T$. W.l.o.g., assume that $s_i < s_j < s_k$ (the case $s_k < s_j < s_i$ can be handled in a similar way). We define the voters’ preferences regarding a_ℓ, b_ℓ, c_ℓ as follows:

1. For each r_x such that $r_x \sqsubset r_i$, set $r_x : a_\ell \succ b_\ell \succ c_\ell$.
2. For each r_y ($y \neq j$) such that $r_i \sqsubset r_y \sqsubset r_k$, set $r_y : b_\ell \succ a_\ell \succ c_\ell$.
3. For each r_z such that $r_k \sqsubset r_z$, set $r_z : c_\ell \succ b_\ell \succ a_\ell$.

After this operation, the profile consists of total orders, and it is clear that it is single-crossing with respect to \sqsubset . \square

The partial orders in the profile from the proof of Theorem 5 are, in fact, weak orders. Thus, we get the next result.

Corollary 6. WO-SCC is NP-complete.

Moreover, we can adapt our proof of Theorem 5 to show that IP-SCC is NP-complete as well.

Theorem 7. IP-SCC problem is NP-complete.

Top Orders The case of top orders is by far the most important and practical one. It turns out that it is also quite challenging: we have not been able to determine the exact complexity of TO-SCC. Nonetheless, we will now describe several polynomial-time algorithms for this problem that work under additional mild constraints on voters’ preferences.

The first of these algorithms (Theorem 8) applies in a situation where we know the complete order of one of the two extreme voters in the profile; alternatively, we could imagine that we define this “extreme” voter ourselves and add it to the profile that we have at hand. In essence, this means that for this algorithm to be applicable, we need to know for every pair of candidates a and b which one of them is closer to a given extreme side of the opinions in the single-crossing spectrum. In practice, in settings where we have a good understanding of the nature of the candidates, we should expect to have the necessary information to use this algorithm.

Theorem 8. There is a polynomial-time algorithm that given an instance $I = (C, \mathcal{R})$ of TO-SCC, where $\mathcal{R} = (r_1, \dots, r_n)$, and an index ℓ such that r_ℓ is a total order, decides if there is an order \sqsubset such that: (i) for each k , $k \neq \ell$, $r_\ell \sqsubset r_k$, and (ii) \mathcal{R} is single-crossing with respect to \sqsubset .

Proof. Without loss of generality, we can assume that $\ell = 1$. Our algorithm consists of two parts. First, in Algorithm \mathcal{L} , we compute an order \sqsubset witnessing that \mathcal{R} is seemingly single-crossing (if indeed it is), and then we invoke Algorithm \mathcal{E} to compute an appropriate extension of \mathcal{R} . If Algorithm \mathcal{L} fails at any point, we reject the input (if we reach Algorithm \mathcal{E} , failure is impossible).

By the theorem’s assumptions, the first element in \sqsubset , r_1 , is a total order. We define a relation \sqsubset^* over $\{r_2, \dots, r_n\}$ as follows: For each i, j , $2 \leq i, j \leq n$, if there is a pair of candidates $a, b \in C$ such that r_1 and r_i order a, b identically but r_j orders them differently, we set $r_i \sqsubset^* r_j$. Algorithm \mathcal{L} is given below:

Algorithm \mathcal{L} : We compute the relation \sqsubset^* over $\{r_2, \dots, r_n\}$ and extend it to relation \sqsubset^{**} over \mathcal{R} as follows: for each pair $i, j \in [n]$ we set $r_i \sqsubset^{**} r_j$ if either $i = 1$ or $r_i \sqsubset^* r_j$. Using the standard algorithm for topological sorting, we check if \sqsubset^{**} can be extended to a linear order. If so, we compute and return this order (this will be our order \sqsubset). If not, we reject.

It is immediate that if this algorithm rejects then \mathcal{R} is not single-crossing with respect to any order \sqsubset that places r_1 first. We claim that if it does not reject, then the profile \mathcal{R} is seemingly single-crossing with respect to the order \sqsubset computed by \mathcal{L} . If it were not, then there would be two candidates a and b and two integers k and ℓ , $1 < k, \ell \leq n$, $k \neq \ell$, such that $r_1 \sqsubset r_k \sqsubset r_\ell$ and $a \succ_{r_1} b$, $b \succ_{r_k} a$, and $a \succ_{r_\ell} b$. However, by definition of \sqsubset^* , we would have $r_\ell \sqsubset^* r_k$, contradicting the fact that Algorithm \mathcal{L} did not reject. Thus, \mathcal{R} is seemingly single-crossing with respect to \sqsubset . Now, by Theorem 4, we can invoke Algorithm \mathcal{E} with \mathcal{R} and \sqsubset as input to get a single-crossing extension of \mathcal{R} . \square

Let u -TO-SCC be the special case of the TO-SCC problem where each vote has at most u unranked candidates. As we can guess the leftmost vote in \sqsubset (n options) and its extensions ($u!$ options), we obtain the following corollary.

Corollary 9. The u -TO-SCC problem can be solved in $\mathcal{O}(2^{u \cdot \log u} \cdot \text{poly}(m, n))$ time.

For our next result, we need to assume that our profile of top orders is *narcissistic*, i.e., every candidate is ranked first by at least one voter; this assumption dates back to the work of Bartholdi and Trick (1986), and has been used in several recent computational social choice papers (Cornaz, Galand, and Spanjaard 2012; Skowron et al. 2013); we expect it to be satisfied when candidates are allowed to vote in the election. For such profiles, we can relax the condition of Theorem 8: we still require that at least one voter submits a total order, but make no assumptions about this voter’s position in the profile. We remark that one can assume that the profile contains a total order if, e.g., the person who wants to understand if the given election is single-crossing is herself a voter

in this election; this assumption is also required for some of the algorithmic results of Lackner (2014), and is known to reduce vote elicitation complexity (Conitzer 2009).

Theorem 10. *There is a polynomial-time algorithm that given an instance $I = (C, \mathcal{R})$ of the TOP PARTIAL ORDER SCC problem, where $\mathcal{R} = (r_1, \dots, r_n)$ is narcissistic and contains at least one total order, decides if \mathcal{R} is single-crossing.*

Independent-Pairs Orders: an Algorithm We can adapt the algorithm from the proof of Theorem 8 to obtain a fixed-parameter tractability result for IP-SCC. Let k -IP-SCC be the special case of IP-SCC where each vote contains at most k incomparable pairs of candidates.

Theorem 11. *There is an algorithm that decides k -IP-SCC in time $\mathcal{O}(2^k \cdot \text{poly}(m, n))$.*

5 Relaxing the Single-Crossing Condition

Throughout this paper, we implicitly assumed that voters’ true preferences are total orders, and the reasons why voters submit partial orders have to do with computation and/or communication constraints. Alternatively, one can imagine that some voters are truly indifferent between certain candidates. It is not clear whether requiring the given profile of partial orders to extend to a single-crossing profile of total orders is the right generalization of the single-crossing condition to such settings. In fact, one can argue that in case of true indifferences seemingly single-crossing profiles are exactly the partial profiles that should be considered single-crossing: indeed, in such profiles no pair of alternatives can be observed to cross more than once. If we view being seemingly single-crossing as a desirable property of a partial profile in its own right, it is natural to ask whether it can be detected efficiently. However, this question turns out to be computationally difficult, even if we restrict ourselves to weak orders or independent-pairs orders.

Proposition 12. *The problem of deciding if a profile of weak orders is seemingly single-crossing is NP-complete. Also, the problem of deciding if a profile of independent-pairs orders is seemingly single-crossing is NP-complete.*

Proposition 12 follows from Corollary 6 and Theorem 7, respectively, by observing that the partial profiles constructed in the respective proofs are single-crossing if and only if they are seemingly single-crossing.

Now, in a seemingly single-crossing profile, as we progress from left to right, for a given pair of candidates a, b we may go from a voter who is indifferent between a and b to one who clearly prefers a to b and then to one who is indifferent between a and b again. It is perhaps more intuitive to require instead that the only allowable transitions are from $a \succ b$ to indifference between a and b to $b \succ a$, or vice versa. We will call such profiles *weakly single-crossing*.

Definition 4. *A profile $\mathcal{R} = (r_1, \dots, r_n)$ of partial orders over a candidate set C is weakly single-crossing with respect to a total order \square over $[n]$ if for every pair of candidates $a, b \in C$ there exist indices $0 \leq k \leq \ell \leq n + 1$ such that either (i) for all $1 \leq i \leq k$ we have $a \succ_{r_i} b$, for all*

$k < i < \ell$ candidates a and b are incomparable in r_i , and for all $\ell \leq i \leq n$ we have $b \succ_{r_i} a$, or, alternatively, (ii) for all $1 \leq i \leq k$ we have $b \succ_{r_i} a$, for all $k < i < \ell$ candidates a and b are incomparable in r_i , and for all $\ell \leq i \leq n$ we have $a \succ_{r_i} b$. A profile $\mathcal{R} = (r_1, \dots, r_n)$ of partial orders is weakly single-crossing if it is weakly single-crossing with respect to some total order \square over $[n]$.

Observe that partial profile \mathcal{R} from Example 1 is not weakly single-crossing with respect to $1 \square 2 \square 3 \square 4$: we go from $a \succ c$ to $a \perp_{r_2} c$ to $a \succ c$. Consequently, the profile \mathcal{R}' from that example is not weakly single-crossing.

Clearly, it is easy to check if a given partial profile \mathcal{R} is weakly single-crossing with respect to a given order \square . Interestingly, while checking whether \mathcal{R} is weakly single-crossing appears to be more difficult, this problem turns out to be polynomial-time solvable as well.

Theorem 13. *There is a polynomial-time algorithm that given a partial profile \mathcal{R} checks whether \mathcal{R} is weakly single-crossing, and, if the answer is positive, outputs an ordering of the voters that witnesses this.*

6 Conclusions and Open Problems

We summarize our results for SCC and SCC-F in Table 1. It is instructive to compare them with recent results of Lackner (2014) and Fitzsimmons (2014) for single-peaked preferences. Lackner proves that one can check in polynomial time whether a profile of partial votes is single-peaked with respect to a given axis. In contrast, verifying the single-crossing property appears to be hard even if the order of the votes is fixed, though we have not been able to obtain a formal hardness result. Moreover, powerful algorithmic techniques that are very useful for working with incomplete single-peaked preferences, such as reductions to 2-SAT and to the consecutive ones problem, while applicable, appear to produce much weaker results in our setting. These are indications that incomplete single-crossing preferences are more difficult to work with than incomplete single-peaked preferences, and new insights are required.

The computational complexity of some of our problems remains open. Perhaps the most intriguing is the complexity of TO-SCC (top orders, arbitrary order of votes) and WO-SCC-F (weak orders, fixed order of votes). Also, given that much of the real-life election data consists of incomplete

	SCC-F	SCC
PO	FPT(a, b) (Cor. 3)	NPc (Thm. 5)
WO	FPT(a, b) (Cor. 3)	NPc (Cor. 6)
TO	P (Thm. 4)	special cases (Thms. 8 and 10), FPT(u) (Cor. 9)
IP	FPT(a, b) (Cor. 3)	NPc (Thm. 7), FPT(k) (Thm. 11)

Table 1: Complexity results: P stands for “polynomial-time solvable”, NPc stands for “NP-complete”, FPT stands for “fixed-parameter tractable”

preference orders, it would be interesting to check often real-life elections admit single-crossing extensions.

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