Proportional Decisions in Perpetual Voting

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Abstract

Perpetual voting is a framework for long-term collective decision making. In this framework, we consider a sequence of subsequent approval-based elections and try to achieve a fair overall outcome. To achieve fairness over time, perpetual voting rules take the history of previous decisions into account and identify voters that were dissatisfied with previous decisions. In this paper, we look at perpetual voting rules from an axiomatic perspective. First, we define two classes of perpetual voting rules that are particularly easy to explain to voters and we explore the bounds imposed by this simplicity. Second, we study proportionality in the perpetual setting and identify two rules with strong proportionality guarantees. However, both rules yield different guarantees and we prove them to be incompatible with each other.

Introduction

In many voting scenarios, a group of voters, for example a committee or working group, has to make several decisions at different points in time. If standard voting rules are used (such as approval, Borda, plurality, etc.), it may happen that a majority dictates all decisions while some voters disagree with every outcome. This can lead to unrepresentative results and, eventually, to dissatisfied voters dropping out of the decision process. Such situations are particularly undesirable if participation in the process is valued highly and if no extreme views are present in the electorate. If a group of colleagues has to regularly agree on a meeting time, it is not acceptable if always the same colleague has to compromise. Similarly, if a committee of volunteers in a sports club is tasked with the organization of a party, no committee member's opinion should be completely ignored.

Perpetual voting, recently introduced by Lackner (2020), is a formalism for tackling these types of long-term decision making processes. From a formal point of view, a perpetual voting instance is a sequence of approval-based elections where each decision has to be made 'online', i.e., in the knowledge of past decisions but without information about future elections. Perpetual voting rules are deterministic, resolute functions that take perpetual voting instances as input and that output a winning alternative for the current decision to be made. Lackner (2020) introduced several perpetual voting rules that aim to achieve a fairer outcome over time as well as basic axioms that formalize desirable properties in the perpetual setting. However, as of now, it was unclear which perpetual rules provide proportional outcomes, i.e., outcomes that reflect the opinions of both large and small groups in a proportional fashion.

Our main goal in this paper is to close this gap and study proportionality in the setting of perpetual voting. This is more difficult than, e.g., proportionality in multi-winner voting (Aziz et al. 2017; Sánchez-Fernández et al. 2017) due to the sequential and dynamic nature of perpetual voting. The main technical difficulty is that voters' preferences (and the set of alternatives) are different each round. Additionally, the online character of perpetual voting prohibits standard methods of (offline) optimization. Despite these obstacles, proportionality is clearly desirable in perpetual voting as it strikes a balance between majoritarian decisions (ignoring minorities) and consensus-based decision (which may result in disproportional power for individuals).

Our starting point, however, is a much more modest desideratum of voting rules: ideally, they should be simple to explain and understand. Thus, we consider two classes of particularly simple perpetual voting rules: win-based and loss-based weighted approval methods (WAMs). Both classes have the advantage of a rather simple definition based on voter weights, which are modified depending on whether a decision was in favour of the voter. Importantly, we require that the magnitude of a change in voter weights is not depending on other voters, i.e., it is always apparent to voters how an outcome influences their weight. We start our analysis by considering two axioms from Lackner (2020): (i) *bounded dry spells* guarantee each voter a satisfying outcome on a regular basis, and (ii) *simple proportionality* is a weak proportionality requirement. Our results show that no voting rule in these two classes can satisfy bounded dry spells. In addition, we characterize all voting rules that satisfy simple proportionality.

This sets the stage for our analysis of proportionality. We introduce two proportionality axioms in the perpetual setting: *lower* and *upper quota for closed groups*. In contrast to simple proportionality, these axioms are applicable in dynamic settings with changing preferences. Our first result is a negative one: while some win-based WAMs satisfy simple proportionality, none of them satisfies one of the stronger properties. Thus, we turn to two more complex perpetual voting rules: Perpetual Consensus (introduced by Lackner 2020) and Perpetual Phragmén (new to the perpetual setting, based on Phragmén 1895). We prove that Perpetual Consensus satisfies the upper quota axiom and Perpetual Phragmén the lower quota axiom. In addition, both rules have bounded dry spells. Finally, we show that Perpetual Phragmén satisfies *perpetual priceability*, an axiom based on work in the multi-winner setting by Peters and Skowron (2020). We prove that this axiom implies the lower quota axiom, but it is incompatible with the upper quota axiom. Thus, we see that Perpetual Phragmén and Perpetual Consensus adhere to two fundamentally incompatible proportionality requirements.

Related work In the last few years, the study of long-term (or repeated) collective decision making has received growing attention. This includes the work of Freeman, Zahedi, and Conitzer (2017), who proposed a sequential mechanism for the aggregation of utility functions over time with the goal to maximize long-term Nash welfare. Variants of this formalism have been studied by Conitzer, Freeman, and Shah (2017) and Freeman et al. (2018). Additionally, Bulteau et al. (2021) studied an offline variant of perpetual voting, focussing on proportionality guarantees achievable in this setting. Notably, this work contains an experimental evaluation of perpetual voting with human participants. Lackner, Maly, and Rey (2021) studied a perpetual version of *participatory budgeting*. Other approaches that consider either temporal aspects of voting or sequences of decisions include *storable votes* (Casella 2005, 2012), *sequential voting rules* (Lang and Xia 2009), *online* approval elections (Do et al. 2022), *Frege's method* (Frege 2000; Harrenstein, Lackner, and Lackner 2020), and *dynamic fair division* (Kash, Procaccia, and Shah 2014; Benade et al. 2018; Zeng and Psomas 2020).

The Perpetual Voting Framework

We will now introduce the perpetual voting formalism, as defined by Lackner (2020), alongside necessary basic definitions. Let $N = \{1, \dots, n\}$ be a set of voters (agents). Given a set of alternatives C, we assume that each voter $v \in N$ approves some non-empty subset of C. An *approval profile* $A = (A(1), \dots, A(n))$ for C is an n-tuple of subsets of C, i.e., $A(v) \subseteq C$ for $v \in N$. We call the triple (N, A, C) a *decision instance*.

A *k-decision sequence* $\mathcal{D} = (N, \bar{A}, \bar{C})$ is a triple consisting of a set of voters N, a k-tuple of sets of alternatives $\bar{C} = (C_1, \dots, C_k)$ and a k-tuple of approval profiles $\bar{A} = (A_1, A_2, \dots, A_k)$ such that A_i is an approval profile for C_i. Thus, for $1 \leqslant i \leqslant k$, the triple (N, A_i, C_i) is a decision instance and can be seen as an individual decision to be made; we refer to it as the *decision instance in round i*.

We write $\bar{w} \in \bar{C}$ as a short hand for $\bar{w} \in \times_{i=1}^{k} C_i$, i.e., $\bar{w} = (w_1, \dots, w_k)$ satisfies $w_i \in C_i$ for $i \in \{1, \dots, k\}$; we refer to $\bar{w}$ as a *k-outcome*. This tuple represents the chosen alternatives in rounds 1 to k. If we combine a k-decision sequence $(N, \bar{A}, \bar{C})$ and a k-outcome $\bar{w} \in \bar{C}$, we speak of a *k-decision history* $\mathcal{H} = (N, \bar{A}, \bar{C}, \bar{w})$, which can be seen as the history of past decision instances alongside the made choices. We thus know, for any $i \leqslant k$, that in case of decision instance (N, A_i, C_i) alternative w_i was chosen.

An important statistic of k-decision histories is the satisfaction of each voter: Given a decision history $\mathcal{H} = (N, \bar{A}, \bar{C}, \bar{w})$, the *satisfaction* of voter $v \in N$ with $\bar{w}$ in round k is $sat_k(v, \bar{w}) = |\{1 \leqslant i \leqslant k : w_i \in A_i(v)\}|$. Thus, the satisfaction of a voter is the number of past decisions that have satisfied this voter. Note that although satisfaction clearly depends on $\mathcal{H}$, we do not explicitly mention that in the notation as $\mathcal{H}$ will always be clear from the context. The same holds for other definitions throughout the paper.

Example 1. *As an example, consider the following 4-decision sequence with four voters $N = \{1, \dots, 4\}$ and four alternatives a, b, c, d (the same in all rounds):*

		voters			
		1	*2*	*3*	*4*
rounds	A_1	$\{a\}$	$\{a\}$	$\{b\}$	$\{c, d\}$
	A_2	$\{a\}$	$\{a, b, c\}$	$\{d\}$	$\{c\}$
	A_3	$\{a\}$	$\{b, c\}$	$\{a, c\}$	$\{b\}$
	A_4	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$

If we assume that we always select the alternative with the highest number of approvals and use alphabetic tie-breaking, then a wins in all rounds. The corresponding 4-outcome is $\bar{w} = (a, a, a, a)$. This means voter 1 is satisfied with every decision ($sat_4(1, \bar{w}) = 4$) while 4 does not agree with any decision ($sat_4(4, \bar{w}) = 0$).

Assume that a group of voters N wants to take a decision and looks back at k decisions already taken. That is, we are presented with a k-decision history $\mathcal{H} = (N, \bar{A}, \bar{C}, \bar{w})$ and a decision instance (N, A_{k+1}, C_{k+1}). The question now is which alternative in C_{k+1} should be chosen, subject to the preferences in A_{k+1} and under consideration of $\mathcal{H}$. An *(approval-based) perpetual voting rule* $\mathcal{R}$ is a function that maps a pair of a decision instance (N, A_{k+1}, C_{k+1}) and a k-decision history $\mathcal{H}$ to an alternative in C_{k+1}.

Given a k-decision sequence $\mathcal{D} = (N, \bar{A}, \bar{C})$, we write $\mathcal{R}(\mathcal{D})$ to denote the k-outcome $\bar{w} \in \bar{C}$ which is selected by applying the perpetual voting rule $\mathcal{R}$ in every round, that is, $\mathcal{R}(\mathcal{D}) = \bar{w}$ is inductively defined by $w_i = \mathcal{R}(N, (A_1, \dots, A_i), (C_1, \dots, C_i), (w_1, \dots, w_{i-1}))$ for $i \leqslant k$. We expect perpetual voting rules to be resolute, i.e., return exactly one winning alternative, therefore we require a tie-breaking order to resolve ties. Throughout the paper, we assume that there exists some arbitrary and fixed order for each set of alternatives that settles ties.

Perpetual Voting Rules

Let us now introduce the perpetual voting rules that we will study in this paper. All of these rules except Perpetual Phragmén have been introduced by Lackner (2020).

First, we consider a natural approach to define perpetual voting rules via weights: voters that have been previously neglected receive a higher weight, voters that are satisfied with previous outcomes receive a lower weight. In each round, the alternative that receives the highest sum of weighted approvals is selected. This idea is captured in a broad sense by the class of *perpetual weighted approval methods*[1] *(WAMs)*, which contains most rules proposed in (Lackner 2020). These approval-based perpetual voting rules are defined as follows: Each voter has an assigned positive weight, which may change each round; a larger weight corresponds to being assigned a higher importance. Let $\alpha_k(v)$ denote voter v's weight in round k. Weights are initialized with $\alpha_1(v) = 1$ for all $v \in N$. The weights of voters in the following rounds are a consequence of the previous history. Formally, there exists a weight function h such that for all $v \in N$, $\alpha_{k+1}(v) = h(v, \mathcal{H})$. Given a k-decision history $\mathcal{H} = (N, \bar{A}, \bar{C}, \bar{w})$ and a decision instance (N, A_{k+1}, C_{k+1}), the rule selects an alternative $w_{k+1} \in C_{k+1}$ with maximum weighted approval score. That is, the score of an alternative c is defined as

$$sc_{k+1}(c) = \sum_{v \in N \text{ with } c \in A_{k+1}(v)} \alpha_{k+1}(v).$$

Observe that WAMs can be computed in polynomial time as long as the function h is computable in polynomial time. This holds for all WAMs considered in this paper.

In this paper, we consider two subclasses subclasses of WAMs: *win-based* and *loss-based WAMs*. These have the benefit of a particularly straight-forward way of calculating weights and thus can be easily explained to voters. For both types, a voter's weight depends only on the voter's weight in the previous round and whether the voter was satisfied with the previous decision. For win-based (loss-based) WAMs, the weights only change for voters who approved (did not approve) a winning alternative. Win-based WAMs can be seen as the perpetual equivalent of the well-known class of the (sequential) Thiele methods used in multi-winner voting (Lackner and Skowron 2023). Similarly, loss-based WAMs are related to dissatisfaction counting rules (Lackner and Skowron 2018).

Definition 1. *We call a WAM* loss-based *if the weights of voters $v \in N$ can be computed as follows:*

$$\alpha_{k+1}(v) = \begin{cases} \alpha_k(v) & \text{if } w_k \in A_k(v), \\ f(\alpha_k(v)) & \text{if } w_k \notin A_k(v), \end{cases}$$

where $f \colon \mathbb{R} \to \mathbb{R}$ is a function satisfying $f(x) \geqslant x$.

We call a WAM win-based *if the weights of voters $v \in N$ can be computed as follows:*

$$\alpha_{k+1}(v) = \begin{cases} g(\alpha_k(v)) & \text{if } w_k \in A_k(v), \\ \alpha_k(v) & \text{if } w_k \notin A_k(v), \end{cases}$$

where $g \colon \mathbb{R} \to \mathbb{R}$ is a function satisfying $0 < g(x) \leqslant x$.

We require $g(x) > 0$ for win-based WAMs as otherwise the voter's weight would remain at 0 since it never increases ($g(x) \leqslant x$ and $f(x) = x$). Observe that, while the function f resp. g can be arbitrarily complex, by definition, the weight of a voter in a win-based WAM only depends on how many rounds she has already won (i.e., how many rounds she was satisfied with). Similarly, the weight of a voter in a loss-based WAM only depends on how many rounds she already lost. In particular, this means that a win- or loss-based WAM is fully defined by an infinite sequence $(G(0), G(1), G(2), \dots)$ such that $G(i)$ is the weight of a voter that has won resp. lost i rounds. In this sense, we believe that win- and loss-based WAMs are simple to explain and understand. The simplest example of a WAM is approval voting (AV), which completely ignores the history of past decisions.

AV. AV is the win-based WAM with $g(x) = x$.

Observe that AV is also a loss-based WAM with $f(x) = x$ and thus the unique rule that is win-based and loss-based. The next method is inspired by Proportional Approval Voting and is thus based on the harmonic series.

Perpetual PAV. Perpetual PAV is a WAM defined by the following weight function:

$$\alpha_{k+1}(v) = \frac{1}{sat_k(v, \bar{w}) + 1} = \begin{cases} \frac{\alpha_k(v)}{\alpha_k(v)+1} & \text{if } w_k \in A_k(v), \\ \alpha_k(v) & \text{if } w_k \notin A_k(v). \end{cases}$$

The last equality shows that Perpetual PAV is indeed a win-based WAM.

An example of a loss-based WAM is Perpetual Unit Cost (Lackner 2020), where the weight of dissatisfied voters is increased by 1.

Perpetual Unit-Cost. Perpetual Unit-Cost is a loss-based WAM defined by $f(x) = x + 1$.

Next, we define two more complicated rules, Perpetual Consensus, introduced by Lackner (2020), and Perpetual Phragmén, a new rule based on Phragmén's sequential rule. As we will see later in Section , both of them can be viewed as proportional—but each in a different sense.

Perpetual Consensus. Let $w_k(v)$ be the weight of voter $v \in N$ in round k. Each voter starts with a weight of $w_0(v) = 1/n$. This WAM is based on the idea that the weight of voters that are satisfied with a decision is reduced in total by 1 and this number is divided equally among them. Consequently, voters can have negative weights[2]; voters with negative weights are not taken into account when determining the winning alternative. After each decision, the weight of all voters is increased by $1/n$. Formally, $N_k^+(c) = \{v \in N : c \in A_k(v) \text{ and } \alpha_k(v) > 0\}$, for all $v \in N$, $\alpha_1(v) = 1/n$ and

$$\alpha_{k+1}(v) = \begin{cases} \alpha_k(v) + \frac{1}{n} - \frac{1}{|N_k^+(w_k)|} & \text{if } w_k \in A_k(v), \\ \alpha_k(v) + \frac{1}{n} & \text{if } w_k \notin A_k(v), \end{cases}$$

Thus, the score of an alternative c is defined as

$$sc_{k+1}(c) = \sum_{v \in N_{k+1}(c)} \max(0, \alpha_{k+1}(v)).$$

Finally, Perpetual Phragmén is a new perpetual rule and is not a WAM. It is inspired by Phragmén's Sequential Rule (Phragmén 1894; Brill et al. 2017).

[1]We note that WAMs in this paper are defined slightly more general than in (Lackner 2020).

[2]Although negative weights are not allowed in the definition of

Perpetual Phragmén. This rule can be described as a load distribution procedure. We assume that winning a round incurs a load of 1, which is distributed to a set of voters that jointly approve the winning alternative. Let $\ell_k(v)$ denote the load assigned to voter v in rounds 1 to k ($\ell_1(v) = 0$). In round $k+1$, for each set of voters N' that jointly approves at least one alternative ($\bigcap_{v \in N'} A_{k+1}(v) \neq \emptyset$), we calculate

$$\ell_{k+1}(N') = \frac{1 + \sum_{v \in N'} \ell_k(v)}{|N'|};$$

this is the load that each of voter in N' would bear if they were selected to choose the winning alternative. We then select a group N' for which $\ell_{k+1}(N')$ is minimal. If more than one set of voters exists with minimal $\ell_{k+1}(N')$, then one of these sets is chosen according to an arbitrary tie-breaking order. Finally, a winning alternative is chosen from $\bigcap_{v \in N'} A_{k+1}(v)$, which is non-empty by the definition of N', according to another arbitrary tie-breaking order. Let N' be the set of voters selected to choose an alternative. Then the loads in the next round are defined as

$$\ell_{k+1}(v) = \begin{cases} \ell_k(v) & \text{if } v \notin N', \\ \ell_{k+1}(N') & \text{if } v \in N'. \end{cases}$$

Conceptually, Perpetual Consensus and Perpetual Phragmén have an important similarity: both are based on distributing a cost (load) of 1 to all voters approving the winning alternative. They differ, however, in the way they distribute this cost. Perpetual Consensus strictly enforces an equal distribution; Perpetual Phragmén assigns a lower load to voters that already have a high load.

Example 2. *Consider the instance from Example 1. In the first round we can distribute the load of a between both of its supporters 1 and 2. This leads to a load distribution where 1 and 2 have load 0.5 while 3 and 4 have load 0. This is clearly better than placing all the load of an alternative on one voter. Hence a wins in round 1. In round 2, both a and c have two supporters. However, due to the higher previous load of the supporters of a, selecting c leads to a more favourable load distribution, where the load of 1 and 3 remains the same at 0.5 and 0 respectively. Moreover, the load of 2 and 4 is set to $\frac{1+0.5}{2} = 0.75$. In round 3, all alternatives have two supporters, but the supporters of a have the lowest previous load, hence it is selected. This leads to a load distribution where all voters have the same load of 0.75. Finally, in round 4 all voters have the same load and all alternatives are supported by exactly one voter. Hence all alternatives would lead to a equally good load distribution. We select some alternative according to a fixed tie-breaking order.*

Proposition 1. *Perpetual Phragmén is not equivalent to any WAM and is computable in polynomial time.*

We note that the Method of Equal Shares (Peters and Skowron 2020), a multi-winner voting rule closely related to Phragmén's Sequential Rule with even stronger proportionality guarantees, has no obvious counter-part in perpetual voting. This is because it requires a priori knowledge about the number of rounds and it is not committee monotone.

WAMs, the definition can easily be adapted to that framework by defining a voting rule that assigns the same weights as Perpetual Consensus if $\alpha_k(v)$ is positive and 0 otherwise. Moreover, observe that compared to Lackner (2020), we divided all weights by n to highlight the similarity to Perpetual Phragmén.

Win- and Loss-Based Voting Rules

First, we want to investigate under which conditions win- and loss-based perpetual voting rules can be proportional. To do so, we first consider *simple proportionality*, a basic axiom proportionality axiom introduced by Lackner (2020).

Simple proportionality considers groups of voters that have identical preferences and guarantees them a *proportional* representation, at least in very simple perpetual voting instances.

Definition 2 (Simple proportionality)**.** *We say that a k-decision sequence $\mathcal{D} = (N, \bar{A}, \bar{C})$ is simple if $A_1 = \cdots = A_k$, $C_1 = \cdots = C_k$, and $|A_1(v)| = 1$ for all $v \in N$. Given a simple decision sequence $\mathcal{D}$ and a voter $v \in N$, let $\#v$ denote the number of voters with identical preferences, i.e., $\#v = |\{v' \in N : A(v') = A(v)\}|$. A perpetual voting rule $\mathcal{R}$ satisfies* simple proportionality *if for any simple n-decision sequence $\mathcal{D}$ with $|N| = n$ it holds that $sat_n(v, \mathcal{R}(\mathcal{D})) = \#v$ for every voter $v \in N$.*

Although this is a quite weak proportionality requirement, similar to weak proportionality in the apportionment setting (Balinski and Young 1982), it is sufficiently strong to reveal that some perpetual voting rules are not proportional. For example, AV fails simple proportionality. On the other hand, Perpetual PAV satisfies simple proportionality (Lackner 2020), witnessing that win-based WAMs can satisfy simple proportionality. Surprisingly, loss-based WAMs are never proportional:

Theorem 2. *There is no loss-based WAM that satisfies simple proportionality.*

Proof. Assume for the sake of a contradiction that $\mathcal{R}$ is a loss-based WAM that satisfies simple proportionality. Now, consider for an arbitrary $k \geqslant 1$ a simple $k+1$-decision sequence $(N, \bar{A}, \bar{C})$ such that $N = \{v_1, \ldots, v_{k+1}\}$, $C_1 = \cdots = C_{k+1} = \{a, b\}$ and v_1 always votes $\{a\}$ and v_i votes $\{b\}$ for all $i \in \{2, \ldots, k+1\}$. Because $\mathcal{R}$ satisfies simple proportionality there are two possible cases in round $k+1$: either a has won one or zero times. In the first case, the score of a in round $k+1$ is $f^{k-1}(1)$ and the score of b is $k \cdot f(1)$ and b must win round $k+1$. Hence,

$$f^{k-1}(1) \leqslant k \cdot f(1) \tag{1}$$

must hold. In the second case,

$$f^{i-1}(1) \leqslant k \text{ for all } i \leqslant k. \tag{2}$$

Now consider a second simple $k+1$-decision sequence $(N, \bar{A}', \bar{C})$ where $\bar{A}'$ is defined by v_1, v_2 always voting $\{a\}$ and $v_3, \ldots, v_{k+1}$ always voting $\{b\}$. Then, there must be a round i where a wins the second time. In this round, the score of b is $(k-1)f(1)$ and the score of a is at most $2 \cdot f^{k-1}(1)$. As we know that a wins in round i we have

$$1/2(k-1)f(1) \leqslant f^{k-1}(1). \tag{3}$$

In summary, we know that for all $k \geqslant 1$ that either $1/2\, kf(1) \leqslant f^k(1) \leqslant (k+1)f(1)$ or $1/2\, kf(1) \leqslant f^k(1) \leqslant k+1$ holds. Consider a simple $2k$-decision sequence $(N'', \bar{A}'', \bar{C}'')$ such that $N = \{v_1, \dots, v_k, w_1, \dots, w_k\}$, $C_1 = \dots = C_{|N|} = \{a, b_1, \dots, b_k\}$, v_i always votes $\{a\}$ and w_i votes $\{b_i\}$ for all $i \in \{1, \dots, k\}$. Furthermore, assume w.l.o.g. that a tie-breaking is applied that always picks b_i over b_j if $i < j$. We claim that b_k does not win any of the $2k$ first rounds. By assumption, for all $i \leqslant k-1$, b_i must win before b_k. Then, the score of a is $k \cdot f^{k-1}(1)$ which is larger than $1/2\, k(k-1)f(1)$ by (3) while the score of b_k is at most $f^{2k-1}(1)$. In the first case we have $f^{2k-1}(1) \leqslant 2kf(1)$ by (1). Clearly, for any k large enough, $2kf(1) < 1/2(k^2-k)f(1)$. Hence, $\mathcal{R}$ does not satisfy simple proportionality. In the second case, we have $f^{2k-1}(1) \leqslant 2k$ by (2). Furthermore, we know $f(1) \geqslant 1$. Now, for any k large enough, $2k < 1/2(k^2-k) \leqslant 1/2(k^2-k)f(1)$. Hence, $\mathcal{R}$ does not satisfy simple proportionality. □

For win-based WAMs, we can precisely characterize which rules satisfy simple proportionality.

Theorem 3. *Let $\mathcal{R}$ be a win-based WAM. Furthermore, define the sequence G as $G(0) = 1$, $G(1) = g(1)$, $G(2) = g(g(1))$, etc. Then, $\mathcal{R}$ satisfies simple proportionality if and only if $xG(x) < (y+1)G(y)$ for all integers $x, y \geqslant 0$.*

Examples of such rules include Perpetual PAV, i.e., $G = (1, 1/2, 1/3, \dots)$, but also, for example, $G = (1, 1/c+1, 1/2c+1, \dots)$ for all $c \geqslant 1$. We see that simple proportionality is satisfied by many win-based WAMs. As we will see in Section , this changes drastically with stronger proportionality axioms.

Moreover, as it turns out, all win- and loss-based WAMs fail another central desideratum of perpetual voting: bounded dry spells. The *bounded dry spells* property guarantees that every voter is satisfied with at least one decision in a bounded number of rounds. This property is very important for creating an incentive to participate in the decision making process.

Definition 3 (Dry spells)**.** *Given a k-decision history $\mathcal{H} = (N, \bar{A}, \bar{C}, \bar{w})$, we say that a voter $v \in N$ has a* dry spell of length ℓ *if there exists $t \leqslant k - \ell$ such that $sat_t(v, \bar{w}) = sat_{t+\ell}(v, \bar{w})$, i.e., voter v is not satisfied with any outcome in rounds $t+1, \dots, t+\ell$.*

Let d be a function from $\mathbb{N}$ to $\mathbb{N}$. A perpetual voting rule $\mathcal{R}$ has a dry spell guarantee of d *if for any decision sequence $\mathcal{D} = (N, \bar{A}, \bar{C})$ and $\bar{w} = \mathcal{R}(\mathcal{D})$, no voter has a dry spell of length $d(|N|)$. A perpetual voting rule $\mathcal{R}$ has* bounded dry spells *if $\mathcal{R}$ has a dry spell guarantee of some d.*

As win- and loss-based WAMs only consider the number of wins—respectively losses—but not the round in which they occur, a long winning streak can be followed by an arbitrarily long dry spell.

Proposition 4. *Every win-based and loss-based WAM has unbounded dry spells.*

In contrast, if we move beyond win- and loss-based WAMs, we find perpetual rules that satisfy both simple proportionality and bounded dry spells. Perpetual Consensus has a dry spell guarantee of at most $1/4 \cdot (n^2 + 3n)$ (Lackner 2020). Here, we show a tight bound for Perpetual Phragmén:

Proposition 5. *Perpetual Phragmén satisfies simple proportionality, and has a dry spell guarantee of $2n-1$ (this bound is tight).*

Proportionality in Perpetual Voting

Simple proportionality, as the name implies, is a very rudimentary notion of proportionality. In particular, it requires identical preferences in all rounds. We will now significantly weaken this assumption and introduce proportionality axioms that are applicable in more dynamic settings.

Let us first define closed groups, which are groups with identical preferences that have no overlapping interests with other voters.

Definition 4. *Given a k-decision sequence $\mathcal{D} = (N, \bar{A}, \bar{C})$, we say that a group $N' \subseteq N$ is* closed *if for every $v \in N'$, $w \in N$, and $i \in \{1, \dots, k\}$ it holds that (i) $A_i(v) = A_i(w)$ if $w \in N'$ and (ii) $A_i(v) \cap A_i(w) = \varnothing$ otherwise.*

The following axioms establish the minimum and maximum influence of a closed group on a decision sequence.

Definition 5 (Perpetual lower/upper quota for closed groups)**.** *A perpetual voting rule $\mathcal{R}$ satisfies* perpetual lower quota for closed groups (LQC) *if, for every k-decision sequence $\mathcal{D}$, it holds for every voter $v \in N$ who is part of a closed group N' that $sat_k(v, \mathcal{R}(\mathcal{D})) \geqslant \left\lfloor k \cdot \frac{|N'|}{n} \right\rfloor$.*

A perpetual voting rule $\mathcal{R}$ satisfies perpetual upper quota for closed groups (UQC) *if, for every k-decision sequence $\mathcal{D}$, it holds for every voter $v \in N$ who is part of a closed group N' that $sat_k(v, \mathcal{R}(\mathcal{D})) \leqslant \left\lceil k \cdot \frac{|N'|}{n} \right\rceil$.*

LQC and UQC identify groups that deserve representation (due to their size and uniformity). These groups should have a roughly proportional influence on the outcome. LQC sets a lower bar for their influence, UQC an upper bar.[3]

Remark 1. *When applying the definitions of LQC and UQC to simple k-decision sequences with $k = n$ (Definition 2), we observe that $\left\lceil k \cdot \frac{|N'|}{n} \right\rceil = \left\lfloor k \cdot \frac{|N'|}{n} \right\rfloor = |N'|$. Thus, both LQC and UQC imply simple proportionality.*

As it turns out, these stronger notions of proportionality cannot be satisfied with a win-based WAM.

Theorem 6. *Every win-based WAM fails both LQC and UQC.*

Proof. Let us first show that every win-based WAM fails UQC. Consider an arbitrary win-based WAM defined by the function g. Now, we construct a simple 2-decision sequence with $2N$ voters. In both rounds, for $i \in \{1, \dots, N\}$ voter i approves c_i, and the voters $N+1, \dots, 2N$ approve alternative c_{N+1}. In round 1, alternative c_{N+1} wins with

[3]We remark that LQC is strictly weaker than perpetual lower quota as introduced by Lackner (2020) (which is too strong to be satisfiable in general). The same holds for UQC and perpetual upper quota (defined analogously).

$sc_1(c_{N+1}) = N$. In the second round, the winning alternative c must be chosen so that $sat_2(v, (c_{N+1}, c)) \leqslant \left\lceil 2 \cdot \frac{\#v}{2N} \right\rceil$ for all $v \in N$. If $v \in \{N+1, \dots, 2N\}$ it holds that $\left\lceil k \cdot \frac{\#v}{n} \right\rceil = 1$, hence c_{N+1} must not win a second time. Consequently $N \cdot g(1) = sc_2(c_{N+1}) < sc_2(c_1) = 1$. As N can be chosen arbitrarily large, we conclude that $g(1) = 0$. This, however, contradicts the definition of win-based WAMs, where $g(x) > 0$ is required.[4]

Now, we show that every win-based WAM fails LQC. Fix a function g defining a win-based WAM. By Theorem 3, we know that $g(1) < g(0) = 1$. Let $k = \left\lceil \frac{1}{g(1)} \right\rceil$. We construct a $k+1$-decision sequence as follows. There are $k+1$ voters. In each round, the set of alternatives is $C = \{c_1, \dots, c_{k+1}\}$. The approval profiles are defined as follows:

	voters				
	1	2	...	k	$k+1$
$A_1 = \dots = A_k$	$\{c_1\}$	$\{c_2\}$	...	$\{c_k\}$	$\{c_{k+1}\}$
A_{k+1}	$\{c_1\}$	$\{c_1\}$	...	$\{c_1\}$	$\{c_{k+1}\}$

We assume that ties are broken in favour of alternatives with smaller index. Thus, in the first round alternative c_1 is winning, in the second round c_2 is winning, etc., until c_k wins in round k. In round $k+1$, alternative c_1 is approved by voters 1 to k and thus $sc_{k+1}(c_1) = k \cdot g(1)$. Further, $sc_{k+1}(c_{k+1}) = g(0) = 1$. Since $sc_{k+1}(c_1) = k \cdot g(1) = \left\lceil \frac{1}{g(1)} \right\rceil \cdot g(1) > 1$, alternative c_1 is winning. This violates LQC since voter $k+1$ is a closed group with satisfaction 0 but deserving a satisfaction of at least $\left\lceil (k+1) \cdot \frac{1}{k+1} \right\rceil = 1$. □

Next, we will show that Perpetual Consensus and Perpetual Phragmén are proportional in a different sense, as the former satisfies UQC while the later satisfies LQC. We begin with Perpetual Consensus.

Theorem 7. *Perpetual Consensus satisfies UQC but fails LQC.*

Before looking at Perpetual Phragmén, we will strengthen LQC by introducing the concept of perpetual priceability. This is motivated by the priceability property from the multi-winner voting setting which was introduced by Peters and Skowron (2020).[5] We first define price systems.

Definition 6. *Given a k-decision sequence $\mathcal{D}$ and an outcome $\bar{w} = (w_1, \dots, w_k)$, we say $\bar{w}$ is supported by the* price system $(B, \{p_i\}_{i \leqslant k})$ *where the real number $B > 0$ is the budget that each voter starts with and for each $i \in \{1, \dots, k\}$, p_i is a function from $N \times C_i$ to $[0, 1]$ such that the following properties hold:*

(P1) $p_i(v, c) = 0$ if $c \notin A_i(v)$, i.e., no voter pays for an alternative that she does not approve.

(P2) $\sum_{j=1}^{k} \sum_{c \in C_j} p_j(v, c) \leqslant B$, i.e., voters cannot spend more than their budget.

(P3) $\sum_{v \in N} p_i(v, w_i) = 1$ for $i \in \{1, \dots, k\}$, i.e., each alternative included in $\bar{w}$ gathers a total payment of 1.[6]

(P4) $\sum_{v \in N} p_i(v, w) = 0$ for $i \in \{1, \dots, k\}$ and $w \neq w_i$, i.e., alternatives not in $\bar{w}$ do not receive any payments.

These four conditions are essentially the same as for priceability in multi-winner voting. In this setting, a fifth condition is present which states that there is no group of voters that supports a common alternative and has a remaining budget of more than 1. Unfortunately, translating this requirement directly into perpetual voting does not work, as the following example shows:

Example 3. *Consider the following 2-decision sequence:*

		voters					
		1	*2*	*3*	*4*	*5*	*6*
rounds	A_1	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$
	A_2	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{b\}$	$\{b\}$

Furthermore, let $\bar{w}$ be an arbitrary outcome for this decision sequence. Assume that there exists a price system $(B, \{p_i\}_{i \leqslant k})$ supporting $\bar{w}$. By construction, the winning alternative in the first round has only one supporter. Therefore, the budget of each voter (B) must be at least 1. However, then after the second round, at least two supporters of the non-winning alternative still have a budget of 1, together strictly more than the price to pay for an alternative (1).

This shows that we need a different minimality condition to use priceability in perpetual voting. Due to the sequential nature of perpetual voting, it turns out that an inductive minimality condition works well.

Definition 7. *Given a k-decision sequence $\mathcal{D}$ and an outcome $\bar{w} = (w_1, \dots, w_k)$, we say $\bar{w}$ is supported by a* minimal price system $(B, \{p_i\}_{i \leqslant k})$ *if there exists a $B^* \leqslant B$ such that the following two conditions hold:*

(P5) there exists a minimal price system $(B^*, \{p_i^*\}_{i \leqslant k-1})$ that supports $(w_1, \dots, w_{k-1})$[7]

(P6) there are no B', w_k' and p_k' such that $B^* \leqslant B' < B$ and $(B', \{p_i\}_{i \leqslant k-1} \cup \{p_k'\})$ is a price system supporting $(w_1, \dots, w_{k-1}, w_k')$.

A perpetual voting rule $\mathcal{R}$ satisfies perpetual priceability *if for any k-decision sequence $\mathcal{D}$ with $\mathcal{R}(\mathcal{D}) = \bar{w}$ there exists a minimal price system $(B, \{p_v\}_{v \in N})$ that supports $\bar{w}$.*

Using this definition, we can show that perpetual priceability implies LQC.

Proposition 8. *Perpetual priceability implies LQC.*

[4] Observe that even if we would change the definition of win-based WAMs to allow g-functions with 0-values, such rules would never satisfy simple proportionality, as follows immediately from Theorem 3 with $x = 1$ and $y = 2$.

[5] Priceability is a rather strong proportionality axiom in the multi-winner setting. It implies Proportional Justified Representation (Sánchez-Fernández et al. 2017) and is incomparable to Extended Justified Representation (Aziz et al. 2017).

[6] Priceability in multi-winner voting is defined slightly differently by fixing the budget of each voter at 1 and varying the price of an alternative. Both definitions are equivalent in the multi-winner setting. The definition with variable budget is better suited for perpetual voting as it allows to extend price-systems to future rounds.

[7] We assume that $(0, \varnothing)$ is a minimal price system supporting the empty sequence.

Proof. Let $\mathcal{R}$ be a perpetual voting rule that satisfies perpetual priceability. We proceed by contradiction. To this end, let $\mathcal{D}$ be a k-decision sequence, such that $\mathcal{R}(\mathcal{D})$ violates LQC and let N' be a closed group that witnesses this violation. Furthermore, let $\mathcal{R}(\mathcal{D}) = (w_1, \ldots, w_k)$ and let $(B, \{p_i\}_{i \leqslant k})$ be a minimal price system that supports $(w_1, \ldots, w_k)$. We first observe that we must have $B \geqslant k/n$, as otherwise it would not be possible to pay for k alternatives. Furthermore, if $B = k/n$, then no budget is left after round k, formally $\sum_{j=1}^{k} \sum_{c \in C_j} p_j(v, c) = B$ for all $v \in N$. As N' is a closed group, we know $A_i(v) = A_i(w)$ for all $v, w \in N'$ and $i \leqslant k$. Furthermore, we know that if $c \notin A_i(v)$, then also $p_i(v, c) = 0$. Therefore, for every voter in $v \in N'$ we have $\sum_{j=1}^{k} \sum_{c \in A_j(c)} p_j(v, c) = |N'| \cdot k/n$. As money can only be spent on alternatives that are elected and every alternative costs 1, this means every voter in N' approves of at least $|N'| \cdot k/n$ alternatives in $\mathcal{R}(\mathcal{D})$ and hence LQC is is satisfied.

Let us now assume $B > k/n$. As we assume that LQC is violated, there is a voter $v \in N'$ such that $sat_k(v, \mathcal{R}(\mathcal{D})) < \left\lfloor k \cdot \frac{|N'|}{n} \right\rfloor$. In particular, that means that only $\left\lfloor k \cdot \frac{|N'|}{n} \right\rfloor - 1$ alternatives supported by the closed group have been elected. Then we know the following for the budget that the voters in N' have left after round k:

$$\sum_{v \in N'} \left(B - \sum_{i \leqslant k} \sum_{c \in C_i} p_i(v, c) \right) \geqslant$$
$$|N'|B - \left(\left\lfloor k \cdot \frac{|N'|}{n} \right\rfloor - 1 \right) >$$
$$|N'|k/n - \left\lfloor k \cdot \frac{|N'|}{n} \right\rfloor + 1 \geqslant 1 \tag{4}$$

Hence there is an ϵ such that the budget of the voters in N' is $1 + \epsilon$. By the definition of minimal price systems, for every $l \leqslant k$, there is a price system $(B_{l-1}, \{p_i^{l-1}\}_{i \leqslant l-1})$ that witnesses the minimality of the price system $(B_l, \{p_i^l\}_{i \leqslant l})$, where $(B_k, \{p_i^k\}_{i \leqslant k}) = (B, \{p_i\}_{i \leqslant k})$.

We claim that $B_1 < B$. Observe that in the first round $B_1 \leqslant 1/|N'|$ must hold, as with a budget of $1/|N'|$ the voters in N' can already afford one of the alternatives that they jointly approve; this would contradict minimality in round 1. Moreover, we can assume that $k \cdot |N'|/n \geqslant 1$ as otherwise LQC is vacously satisfied. It follows that $k \geqslant n/|N'|$. Finally, by assumption $B > k/n$. Put together, we have

$$B_1 \leqslant \frac{1}{|N'|} = \frac{n}{n|N'|} \leqslant \frac{k}{n} < B.$$

Now let l^* be the largest index l for which $B_l < B$. We claim that there is a B' with $B_{l*} \leqslant B' < B_{l*+1} = B$ such that there are $w' \in C_{l*+1}$ and p'_{l*+1} such that $(B', \{p_i^{l*}\}_{i \leqslant l*} \cup \{p'_{l*+1}\})$ is a price system supporting $(w_1, \ldots, w_{l*}, w')$. This would be a contradiction to the minimality of $(B_{l*+1}, \{p_i^{l*+1}\}_{i \leqslant l*+1})$.

Let w' be an alternative supported by the voters in the closed group in round $l^* + 1$. Furthermore, let $B' = \max(B - \epsilon/|N'|, B_{l*})$. Observe that $B_{l*} \leqslant B' < B$. Now, define p'_{l*+1} such that $\sum_{v \in N'} p'_{l*+1}(v, w') = 1$ and $p'_{l*+1}(v', c) = 0$ whenever $v' \notin N'$ or $c \neq w'$. This is possible, because the voters in N' have at least a budget of 1 in round $l^* + 1$. Hence this is a price system that supports $(w_1, \ldots, w_{k-1}, w')$. Contradiction to the minimality of $(B, \{p_i\}_{i \leqslant k})$. ☐

	SP	BD	LQC	UQC
AV	×⋆	∞⋆	×	×
Per. Unit-Cost	×⋆	∞⋆	×	×
Per. PAV	✓⋆	∞⋆	×	×
Per. Consensus	✓⋆	$\leqslant \frac{n^2+3n}{4}$ ⋆	×	✓
Per. Phragmén	✓	$2n - 1$	✓	×

Table 1: Axiomatic results for selected perpetual voting rules: bounded dry spells (BD), simple proportionality (SP), and lower/upper quota for closed groups (LQC/UQC). Entries marked with ⋆ are due to Lackner (2020).

It can be shown that one can always turn the load balancing procedure of Perpetual Phragmén into a minimal price system. Therefore it satisfies perpetual priceability.

Proposition 9. *Perpetual Phragmén satisfies perpetual priceability.*

Finally, we observe that perpetual priceability is incompatible with UQC.

Proposition 10. *A perpetual voting rule cannot satisfy both perpetual priceability and UQC.*

Corollary 11. *Perpetual Phragmén satisfies LQC but fails UQC.*

Discussion and Research Directions

We provide a summary of our axiomatic results in Table 1. Two rules appear to be most promising: Perpetual Consensus and Perpetual Phragmén. Their most notable difference is in which sense they are proportional: Perpetual Phragmén satisfies a lower quota axiom (guaranteeing groups a certain satisfaction), whereas Perpetual Consensus satisfies an upper quota axiom (limiting excessive influence of groups on the decision process). Moreover, we have seen that the simplicity of win- and loss-based WAMs is too restrictive to achieve proportional outcomes. Perpetual Consensus or Perpetual Phragmén are more proportional but also conceptually more difficult. Whether the conceptual complexity of these rules is problematic can only be answered in the context of a concrete application.

A natural open question is whether a voting rule exists that satisfies both LQC and UQC. The Quota apportionment method of Balinski and Young (1975) may be a useful starting point. Note that such a rule cannot satisfy perpetual priceability (Proposition 10). Currently, Perpetual Phragmén is the only perpetual voting rule satisfying perpetual priceability. Another candidate for this property is an an adaption of the minimax support method (Fernández et al. 2022) to the perpetual setting.

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Proof Details from Section

Proposition 1. *Perpetual Phragmén is not equivalent to any WAM and is computable in polynomial time.*

Proof. To prove the first statement, we consider a decision sequence with 7 voters. In the first two rounds, the preferences are

$$A_1 = A_2 = (\{a\}, \{a\}, \{a\}, \{b\}, \{b\}, \{b\}, \{c\}).$$

Thus, a wins in the first round (we assume alphabetic tiebreaking) and the corresponding loads are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0)$. In the second round b wins and the loads are $(\frac{1}{3}, \dots, \frac{1}{3}, 0)$. Assume towards a contradiction that Perpetual Phragmén is a WAM. We can thus assign weights in some fashion; let these be $x_1, \dots, x_7$.

We now consider several decision instances for round three. First, if the preferences are

$$A_3 = (\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{a, b, c, d, e, f\}),$$

then all alternatives are tied. Thus, we can conclude that $x_1 = x_2 = x_3 = x_4 = x_5 = x_6$; let $x = x_1 = \dots = x_6$ and $y = x_7$. Second, if the preferences are

$$A'_3 = (\{a\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}),$$

then a wins (the load of voter 1 and 2 would increase to $\frac{5}{6}$). Thus, we infer that $2x > y$. Finally, we consider

$$A''_3 = (\{a\}, \{a\}, \{a\}, \{b\}, \{c\}, \{c\}).$$

Here a and c are tied; in both cases the load of the corresponding voters (voters $1, 2, 3$ for a, voters $5, 6$ for c) would increase to $\frac{2}{3}$. Thus, it holds that $3x = x + y$ and in turn $2x = y$. This contradicts our previous result that $2x > y$. We conclude that Perpetual Phragmén cannot be "simulated" by a WAM.

Next, we show that Perpetual Phragmén is computable in polynomial time. To calculate $\ell_{k+1}(v)$ for all $v \in N$, we have to first find the set of voters N' for which $\ell_{k+1}(N')$ is minimal. Recall that

$$\ell_{k+1}(N') = \frac{1 + \sum_{v \in N'} \ell_k(v)}{|N'|}.$$

Let $N(c) = \{v \in N : A(v) = \{c\}\}$ for $c \in C$. We will calculate for each alternative $c \in C_{k+1}$ the subset $N'(c) \subseteq N(c)$ that has a minimal $\ell_{k+1}(N'(c))$ among all subsets of $N(c)$. To do this we sort voters in $N(c)$ by their load: $\ell_k(v_1) < \ell_k(v_2) < \dots < \ell_k(v_s)$ with $N(c) = \{v_1, \dots, v_s\}$.

We claim that $N'(c)$ is an interval containing v_1 in this order, i.e., there exists a $t \leqslant s$ such that $N'(c) = \{v_1, \dots, v_t\}$. Towards a contradiction, assume that $N'(c)$ does contain v_t but not v_r with $r < t$. Then clearly replacing $\ell_{k+1}(N'(c)) > \ell_{k+1}(N'(c) + v_t - v_r)$; a contradiction.

Since $N'(c)$ consists of an interval where voters are sorted by load, we can determine the optimal value by adding voters one by one (starting with the lowest-load voter, i.e., v_1). Then, we compare $N'(c)$ for all $c \in C_{k+1}$ and thus find N' with minimal $\ell_{k+1}(N')$. This procedure requires polynomial time. □

Proof Details from Section

First, observe that win- and loss-based WAMs have the following very useful property:

Lemma 12. *Let $\mathcal{R}$ be a win-based or a loss-based WAM and let $(N, \bar{A}, \bar{C})$ and $(N^*, \bar{A}^*, \bar{C}^*)$ be two k-decision sequences such that and $w_i = w_i^*$ for all i under $\mathcal{R}$. Then for all voters $v \in N$ the weight of v after round k is the same for the decision sequences $(N, \bar{A}, \bar{C})$ and $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$. Here $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$ denotes the decision sequence with alternatives $C \cup C^*$, $N + N^*$ voters where the first N voters vote like the voters in $(N, \bar{A}, \bar{C})$ and the voters from $N + 1$ to $N + N^*$ vote like the voters in $(N^*, \bar{A}^*, \bar{C}^*)$.*

Proof. We prove the lemma by induction over k. By definition the weight of all voters in the first round is 1. Now assume that the weights of all voters are the same in $(N, \bar{A}, \bar{C})$ resp. $(N^*, \bar{A}^*, \bar{C}^*)$ and $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$ in round $k - 1$. Let w be the winner in round $k - 1$ for $(N, \bar{A}, \bar{C})$ and $(N^*, \bar{A}^*, \bar{C}^*)$. Then, for every alternative $c \in C_{k-1} \cup C^*_{k+1}$ we have $sc_{k-1}(w) > sc_{k-1}(c)$ in $(N, \bar{A}, \bar{C})$ and $(N^*, \bar{A}^*, \bar{C}^*)$ (if $c \notin C_{k-1}$, then we set $sc_{k-1}(c) = 0$ in $(N, \bar{A}, \bar{C})$ and the same for $(N^*, \bar{A}^*, \bar{C}^*)$.) As the weigth of all voters is the same in $(N, \bar{A}, \bar{C})$ resp. $(N^*, \bar{A}^*, \bar{C}^*)$ and $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$ in round $k - 1$, the score of all alternatives in $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$ is just the sum of their scores in $(N, \bar{A}, \bar{C})$ and $(N^*, \bar{A}^*, \bar{C}^*)$. Therefore, we have $sc_{k-1}(w) > sc_{k-1}(c)$ in $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$ for all $c \in C_{k-1} \cup C^*_{k-1}$, hence w is the winner in round $k - 1$. However, then the weight of any voter v in round k is either

$$\alpha_k(v) = \begin{cases} \alpha_{k-1}(v) & \text{if } w \notin A_{k-1}(v), \\ g(\alpha_{k-1}(v)) & \text{if } w \in A_{k-1}(v). \end{cases}$$

or

$$\alpha_k(v) = \begin{cases} f(\alpha_{k-1}(v)) & \text{if } w \notin A_{k-1}(v), \\ \alpha_{k-1}(v) & \text{if } w \in A_{k-1}(v). \end{cases}$$

where α_{k-1}, $A_{k-1}(v)$ and w are the same for $(N, \bar{A}, \bar{C})$ and $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$. Hence, $\alpha_k(v)$ is the same in $(N, \bar{A}, \bar{C})$ and $(N + N^*, \bar{A} + \bar{A}^*, \bar{C} + \bar{C}^*)$ for all $v \in N$. □

Theorem 3. *Let $\mathcal{R}$ be a win-based WAM. Furthermore, define the sequence G as $G(0) = 1$, $G(1) = g(1)$, $G(2) = g(g(1))$, etc. Then, $\mathcal{R}$ satisfies simple proportionality if and only if $xG(x) < (y + 1)G(y)$ for all integers $x, y \geqslant 0$.*

Proof. Assume that $xG(x) < (y + 1)G(y)$ for all integers $x, y \geqslant 0$. If we set $y = x - 1$, we obtain

$$G(x) < G(x - 1) \tag{5}$$

Towards a contradiction, assume that the rule fails simple proportionality for some simple $|N|$-decision sequence $(N, \bar{A}, \bar{C})$ and corresponding $|N|$-outcome $\bar{w}$. Now, let $k + 1$ be the first round such that there is a voter v with $sat_{k+1}(v, \bar{w}) = \#v + 1$. Such a round must exist, because simple proportionality is violated. There also exists a voter v' with $sat_{k+1}(v', \bar{w}) < \#v'$. Let $\#v = x$ and $\#v' = y + 1$. Observe that, by assumption,

$sat_k(v, \bar{w}) = \#v$. As v wins in round $k+1$, it holds that $xG(sat_k(v, \bar{w})) = xG(x) \geqslant (y+1) \cdot G(sat_k(v', \bar{w}))$. Since $sat_k(v', \bar{w}) = sat_{k+1}(v', \bar{w}) < \#v' = y+1$, by (5) we have $G(sat_k(v', \bar{w})) \geqslant G(y)$. Thus, $xG(x) \geqslant y \cdot G(sat_k(v', \bar{w})) \geqslant (y+1)G(y)$, a contradiction.

For the other direction, assume that simple proportionality holds. First, let us show that $G(x) < G(x-1)$. Consider a simple $(2x)$-simple profile with x voters approving some alternative and the remaining x voters approving some other alternative. In round $2x$, one of these groups has satisfaction $x-1$, the other x. By simple proportionality, the group with satisfaction $x-1$ must win (independent of tiebreaking). Thus, $xG(x) < xG(x-1)$, i.e., $G(x) < G(x-1)$.

Next, we show that $xG(x) < (y+1)G(w_y)$. Consider an $(x+y+1)$-simple profile with two groups: x voters approving one alternative, $y+1$ voters approve another alternative. Consider the round k where the latter group wins for the $(y+1)$-st time (this has to happen due to simple proportionality). Let $x' \leqslant x$ be the satisfaction of the former group in round k. Assuming that tiebreaking is against this group, it holds that $(y+1)G(y) > xG(x') \geqslant xG(x)$. □

Proposition 4. *Every win-based and loss-based WAM has unbounded dry spells.*

Proof. Let $\mathcal{R}$ be either a win-based or a loss-based WAM, i.e., either $g(x) = x$ or $f(x) = x$ holds. We assume for the sake of a contradiction that there is a function d that bounds the dry spells of $\mathcal{R}$.

First, assume $g(x) = x$ and there is a value x^* (that can appear) such that $f(x^*) = x^*$ [$f(x) = x$ and there is a value x^* (that can appear) such that $g(x^*) = x^*$]. Then, let $(N, \bar{A}, \bar{C})$ be a k-decision sequence such that there is a voter that has weight x^* in round $k+1$. We consider the decision sequence that consist of three copies of $(N, \bar{A}, \bar{C})$. Then, by Lemma 12 there are three voters v_1, v_2, v_3 with weight x^* in round $k+1$. Now, consider that in the next $d(3|N|)+1$ rounds there are two alternatives $\{a, b\}$, voter v_1 votes for an alternative a, voter v_2 and v_3 vote b and every other voter votes $\{a, b\}$. Then the score of alternative a is always $x^* + V$, where V is the weight of the voters in $V \backslash \{v_1, v_2, v_3\}$, and the score of alternative b is always $2x^* + V$. Therefore, b has a higher score in every election and v_1 has a dry spell of $d(3|N|)+1$ rounds. A contradiction.

Now, assume that $g(x) = x$ and $f(x) > x$ [or $f(x) = x$ and $g(x) < x$] on all values x that can appear as weights of voters. Now consider an election with two voters v_1 and v_2, two alternatives a and b and $2d(3)+3$ rounds, where v_1 always votes a and v_2 always votes b. Then, either v_1 or v_2 looses at least $d(3)+2$ rounds. We assume w.l.o.g. that v_1 looses $d(3)+2$ rounds. Then, the weight of voter v_1 in round $2d(3)+4$ is $f^{d(3)+2}(1)$ [$g^{d(3)+1}(1)$]. Now, we add another voter v_3 that votes $\{a, b\}$ in the first $2d(3)+3$ rounds. Hence, she wins in every round and his weight in round $2d(3)+4$ is 1 [$g^{2d(3)+3}(1)$]. Then, assume that in the next $d(3)+1$ rounds voters v_1 and v_2 vote a and voter v_3 votes b. We claim that voter v_3 wins non of these $d(3)+1$ rounds: Alternative a always has a score of at least $f^{d(3)+2}(1)$ [$g^{2d(3)+2}(1)$], whereas alternative b has a score of at most $f^{d(3)+1}(1)$ [$g^{2d(3)+3}(1)$]. As $f(x) > x$ [$g(x) < x$], alternative a has a higher score in every election. Therefore, v_3 has a dry spell of $d(3)+1$, a contradiction. □

Proposition 5. *Perpetual Phragmén satisfies simple proportionality, and has a dry spell guarantee of* $2n-1$ *(this bound is tight).*

Proof.
1. Simple proportionality follows immediately from the fact that Phragmén's sequential rule (the approval-based multi-winner rule) behaves like the D'Hondt method in the apportionment setting (see, e.g., (Brill, Laslier, and Skowron 2017)).
2. First, observe that the minimum load and maximum load of voters cannot differ by more than 1; otherwise a different outcome (in favour of those with a smaller load) would have been chosen previously. Further, each round the total load increases by 1. Towards a contradiction, assume that a voter v is not satisfied with a sequence of $2n-1$ decisions. Let α be the load of voter v during these $2n-1$ rounds. Further, let $N' = N \backslash \{v\}$. The total load the voters in N' is at least $(n-1)(\alpha-1)$ before these rounds, and at least $(n-1)(\alpha-1) + 2n - 1$ after these $2n-1$ rounds. Thus, the average load of voters in N' is at least $\alpha + 1 + \frac{1}{n-1}$. Hence, there is one voter with a load strictly larger than $\alpha+1$, a contradiction. Hence, we obtain a dry spell guarantee of $2n-1$.
   To see that this bound is tight, consider the following decision sequence with n voters. All voters have disjoint approval sets. In round 1, tiebreaking in favour of voter n's alternative, in later rounds it is always against voter n. Voter n loses in rounds $2, \dots, 2n-1$, but wins in round $2n$. This is a dry spell of length $2n-2$.

□

Proof Details from Section

Theorem 7. *Perpetual Consensus satisfies UQC but fails LQC.*

Proof. To see that Perpetual Consensus satisfies UQC, let $\mathcal{D} = (N, \bar{A}, \bar{C})$ be a k-decision sequence and N' be a closed group. Assume towards a contradiction that $\mathcal{D}$ witnesses that Perpetual Consensus fails UQC and that k is the shortest decision sequence for which this can happen. Consequently, for $v \in N'$, $sat_k(v, \mathcal{R}(\mathcal{D})) > \left\lceil k \cdot \frac{|N'|}{n} \right\rceil$. We can assume without loss of generality that v was satisfied with the decision in round k (i.e., $sat_k(v, \mathcal{R}(\mathcal{D})) = sat_{k-1}(v, \mathcal{R}(\mathcal{D})) + 1$), because otherwise we could consider a $(k-1)$-decision sequence witnessing that Perpetual Consensus fails UQC. Let $x = \left\lceil k \cdot \frac{|N'|}{n} \right\rceil$. Thus, $sat_k(v, \mathcal{R}(\mathcal{D})) = x + 1$. Let us calculate the weight of v in round k:

$$\alpha_k(v) = \underbrace{1 + \cdots + 1}_{k \text{ rounds}} - \underbrace{\frac{x \cdot n}{|N'|}}_{v \text{ has won } x \text{ times}} \leqslant k - k = 0.$$

We see that for $v \in N'$, $\alpha_k(v) = 0$ and hence N' has a total weight of $\leqslant 0$ in round k. As N' is a closed group, no alternatives approved by a voter in N' will be chosen (some other alternative will have a positive weight). This contradicts our assumption that the satisfaction of voters in N' increases in round k.

To show that Perpetual Consensus fails LQC, we construct a k-decision sequence with five closed groups. The five groups consist of 1, 3, 5, 18, and 27 voters, respectively. These are all voters, 54 in total. By definition of a closed group, each group has distinct alternatives they approve. We consider $k = 30$. Perpetual consensus produces an outcome such that alternatives from the first group win once, from the second group twice, from the third group three times, from the fourth ten times, and from the fifth group 14 times. We see that produces the apportionment $\mathbf{a} = (1, 2, 3, 10, 14)$. For voters v in the fifth group we have $sat_{30}(v, \mathcal{R}(\mathcal{D})) = 14 < \lfloor 30 \cdot \frac{15}{30} \rfloor$, thus violating LQC. □

Proposition 9. *Perpetual Phragmén satisfies perpetual priceability.*

Proof. We show the result by proving that the load distribution produced by Perpetual Phragmén can be turned into a minimal price system supporting the outcome of Perpetual Phragmén. To this end, let $\mathcal{D}$ be a k-decision sequence, let $\bar{w} = (w_1, \dots, w_k)$ be the outcome of Perpetual Phragmén for $\mathcal{D}$ and let $\ell_1, \dots, \ell_k$ be the load distribution produced by Perpetual Phragmén. Then we claim that the following is a minimal price system that supports $\bar{w}$:

$$B = \max_{v \in N}(\ell_k(v))$$

and, for all $i \leqslant k$,

$$p_i(v, c) = \ell_i(v) - \ell(v)_{i-1}.$$

First we show that $(B, \{p_i\}_{i \leqslant k})$ is a price system. For (P1), we observe that in any round i the load of a voter v that does not approve w_i cannot increase. Hence, $p_i(v, w_i) = \ell_i(v) - \ell_{i-1}(v) = \ell_{i-1}(v) - \ell_{i-1}(v) = 0$. For (P2), we note that for all v

$$\sum_{j=1}^{k} \sum_{c \in C_j} p_j(v, c) = \sum_{j=1}^{k} \ell_i(v) - \ell(v)_{i-1} = \ell_k(v) - \ell_0(v) = \ell_k(v) \leqslant B.$$

Now consider (P3). Consider round i and let N' be the set of voters v for which $\ell_i(v) > \ell_{i-1}(v)$ holds. Then, by definition of Perpetual Phragmén, for all $v \in N'$ we have

$$\ell_i(N') = \frac{1 + \sum_{v \in N'} \ell_{i-1}(v)}{|N'|}.$$

Therefore,

$$\sum_{v \in N} p_i(v, w_i) = \sum_{v \in N'} \ell_i(v) - \ell_{i-1}(v) = |N'| \frac{1 + \sum_{v \in N'} \ell_{i-1}(v)}{|N'|} - \sum_{v \in N'} \ell_{i-1}(v) = 1.$$

Finally, (P4) holds by definition.

It remains to show that $(B, \{p_i\}_{i \leqslant k})$ is a minimal price system. For this, we proceed by induction. First, assume $\mathcal{D}$ is a 1-decision sequence. For the sake of a contradiction, assume (B', p'_1) is a price system supporting (w') with $B' < B$. Let B' be the minimal budget for which such a price system exists. Then, we can assume that for all $v \in N$ we have either $p'_1(v, w') = 0$ or $p'_1(v, w') = B'$. Thus, $\ell'_1(v) = p'_1(v, w')$ is a valid load distribution and $\ell'_1(v) < \ell_1(v)$. This contradicts the assumption that ℓ_1 was the load distribution chosen by Perpetual Phragmén.

Now, assume that for all $(k-1)$-decision sequences the price system generated from the load distribution is minimal and consider a k decision sequence $\mathcal{D}$. Let $(B, \{p_i^*\}_{i \leqslant k})$ be the price system generated by running Perpetual Phragmén on $\mathcal{D}$ and let $(B^*, \{p_i^*\}_{i \leqslant k-1})$ be the price system generated by running Perpetual Phragmén on the first $k-1$ rounds of $\mathcal{D}$. By construction $p_i = p_i^*$ for all $i \leqslant k-1$. Furthermore, we know by the induction hypothesis that $(B^*, \{p_i^*\}_{i \leqslant k-1})$ is a minimal price system.

First, assume $B^* = B$. In that case $(B, \{p_i\}_{i \leqslant k})$ is also trivially a minimal price system. Now assume $B^* < B$. For the sake of a contradiction, assume that there are B', w'_k and p'_k such that $B^* \leqslant B' < B$ and $(B', \{p_i\}_{i \leqslant k-1} \cup \{p'_k\})$ is a price system supporting $(w_1, \dots, w_{k-1}, w'_k)$. Let B' the minimal budget for which this is possible. Let us first assume that $B^* = B'$. Observe that for all $v \in N$ we have $\sum_{i \leqslant k-1} \sum_{c \in C_i} p_i(v, c) = \ell_{k-1}(v)$ by construction. Then, by (P1), we know

$$p'_k(v, w') + \sum_{i \leqslant k-1} \sum_{c \in C_i} p_i(v, c) = p'_k(v, w') + \ell_{k-1}(v) \leqslant B' = \max_{v \in N}(\ell_{k-1}(v)).$$

As $\sum_{v \in N} p'_k(v, w') = 1$ this implies that there is a load distribution ℓ'_k such that $\max_{v \in N}(\ell'_k(v)) = \max_{v \in N}(\ell_{k-1}(v)) = B^* < B = \max_{v \in N}(\ell_k(v))$. However, this contradicts the assumption that ℓ_k was chosen as a load distribution by Perpetual Phragmén.

Now assume $B^* < B'$. We claim that we can turn p'_k into a load distribution ℓ'_k with lower load than ℓ_k as follows. For all $v \in N$ we have:

$$\ell'_k(v) = p'_k(v, w') + \sum_{i \leqslant k-1} \sum_{c \in C_i} p_i(v, c).$$

In particular, that means $\ell_{k-1}(v) = \ell'_k(v)$ for all v such that $p'_k(v, w') = 0$. Let $N' = \{v \mid p'_k(v, w') \neq 0\}$. We claim that for all $v \in N'$ we have $\ell'_k(v) = B'$. Otherwise, we could construct a price system with lower budget as before. From this, we get

$$\begin{aligned} |N'|\ell'_k(v) &= \sum_{v \in N'} \ell'_k(v) \\ &= \sum_{v \in N'} \left(p'_k(v, w') + \sum_{i \leqslant k-1} \sum_{c \in C_i} p_i(v, c) \right) \\ &= \sum_{v \in N'} \left(p'_k(v, w') + \ell_{k-1}(v) \right) \\ &= 1 + \sum_{v \in N'} \ell_{k-1}(v) \end{aligned}$$

and hence $\ell'_k(v) = \frac{1+\sum_{v\in N'} \ell_{k-1}(v)}{|N'|}$. This shows that $\ell'_k(v)$ is indeed a valid load distribution. As $B' < B$, ℓ'_k is a better load distribution than ℓ_k. This is a contradiction to the assumption that Perpetual Phragmén has chosen load distribution ℓ_k. □

Proposition 10. *A perpetual voting rule cannot satisfy both perpetual priceability and UQC.*

Proof. Consider the following 2-decision sequence. Let $N = \{1, \dots, 20\}$, $C_1 = C_2 = \{c, d_1, \dots, d_{10}\}$. Further let

$$A_1(v) = \begin{cases} \{d_v\} & \text{if } v \in \{1, \dots, 10\}, \\ \{c\} & \text{if } v \in \{11, \dots, 20\}. \end{cases}$$

and $A_2 = A_1$. Assume towards a contradiction that $\mathcal{R}$ is a perpetual voting rule satisfying both perpetual priceability and UQC. Let $\mathcal{R}(\mathcal{D}) = (w_1, w_2)$. Towards a contradiction, assume that $w_1 \in \{d_1, \dots, d_{10}\}$. As $d_1, \dots, d_{10}$ is approved by only one voter each and $\sum_{v\in N} p_v(1, w_1) = 1$ (Condition P3), we know that $B \geqslant 1$ (Condition P2). This violates (P6), since the alternative c can be supported by a price system with $B' = 0.1$. Thus, $w_1 = c$. Now, in round 2, assume again that $w_2 \in \{d_1, \dots, d_{10}\}$. By the same argument as before, we infer that $B \geqslant 1$. This violates (P6), since the outcome (c, c) can be supported by a price system with $B' = 0.2$. Thus, $\mathcal{R}(\mathcal{D}) = (c, c)$. Now, note that $N' = \{11, \dots, 20\}$ is a closed group. Hence, for $v \in N'$, UQC implies that $sat_2(v, \mathcal{R}(\mathcal{D})) \leqslant \left\lceil k \cdot \frac{|N'|}{n} \right\rceil = 1$. However, $sat_2(v, (c, c)) = 2$, a contradiction. □