



Consistent approval-based multi-winner rules

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Abstract

This paper is an axiomatic study of consistent approval-based committee (ABC) rules. These are multi-winner voting rules that select a committee, i.e., a fixed-size group of candidates, based on approval ballots. We introduce the class of ABC scoring rules and provide an axiomatic characterization of this class based on the consistency axiom. Building upon this result, we axiomatically characterize three important consistent multi-winner rules: Proportional Approval Voting, Multi-Winner Approval Voting and the Approval Chamberlin–Courant rule. Our results demonstrate the variety of ABC scoring rules and illustrate three different, orthogonal principles that multi-winner voting rules may represent: proportionality, diversity, and individual excellence.

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1. Introduction

A *multi-winner* rule selects a fixed-size set of candidates—a *committee*—based on the preferences of voters. Multi-winner elections are of importance in a wide range of scenarios, which often fit in, but are not limited to, one of the following three categories (Elkind et al., 2017; Faliszewski et al., 2017). The first category contains multi-winner elections aiming for *propor-*

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tionality. The archetypal example of a multi-winner election is that of selecting a representative body such as a parliament. The second category comprises multi-winner elections with the goal that as many voters as possible should have an acceptable representative in the committee. Consequently, there is no or little weight put on giving voters a second representative in the committee. This goal may be desirable, e.g., in a deliberative democracy (Chamberlin and Courant, 1983; Dryzek and List, 2003). Voting rules suitable in such scenarios follow the principle of *diversity*. The third category contains scenarios where the goal is to choose a fixed number of best candidates and where ballots are viewed as expert judgments. Here, the chosen multi-winner rule should follow the *individual excellence* principle. An example is shortlisting nominees for an award where a nomination itself is often viewed as an achievement.

We consider multi-winner rules based on approval ballots, which allow voters to express *di-chotomous preferences*. An approval ballot thus corresponds to a subset of (approved) candidates. A simple example of an approval-based election can highlight the distinct nature of proportionality, diversity, and individual excellence: There are 100 voters and 5 candidates $\{a, b, c, d, e\}$: 66 voters approve the set $\{a, b, c\}$, 33 voters approve $\{d\}$, and one voter approves $\{e\}$. Assume we want to select a committee of size three. If we follow the principle of proportionality, we could choose, e.g., $\{a, b, d\}$; this committee closely reflects the proportions of voter support. If we aim for diversity and do not consider it important to give voters more than one representative, we may choose the committee $\{a, d, e\}$: it contains one approved candidate of every voter. The principle of individual excellence aims to select the strongest candidates: a, b , and c have most supporters and are thus a natural choice, although the opinions of 34 voters are essentially ignored. We see that these three intuitive principles give rise to very different committees.

It is relatively easy to explain what proportionality, diversity, and individual excellence means when the voters' preferences have a specific structure as in the above example. There, for any two voters, their approval sets are either the same or disjoint; this is equivalent to saying that the voters and candidates can be divided into disjoint groups so that each group of voters approves a single group of candidates (intuitively, such a group of candidates can be viewed as a political party)—we thus call such preference profiles *party-list*. However, specifying how a multiwinner voting rule should act on party-list profiles does not—on its own—provide comprehensive guidance for choosing committees for the general model. To achieve that, one needs to rely on more general principles. Our analysis is thus based on four basic principles (framed as axioms):

- (i) symmetry: the identity of voters and candidates should not affect the result of an election,
- (ii) consistency¹: if two disjoint societies both collectively prefer committee W_1 to committee W_2 , then the union of these two societies should also collectively prefer W_1 to W_2 ,
- (iii) weak efficiency: a committee should contain approved candidates rather than candidates that are not approved by anyone,
- (iv) continuity: sufficiently large majorities should be able to dictate the decision.

As we show in this paper, among the rules that satisfy consistency, symmetry, continuity, and weak efficiency, specifying what proportionality means for party-list profiles is sufficient to characterize a unique multi-winner rule. The same holds for diversity and individual excellence.

¹ This is a straightforward adaption of consistency as defined for single-winner rules by Smith (1973) and Young (1974a).

In this work, we focus on approval-based committee (ABC) ranking rules, i.e., rules that produce a ranking of all committees, rather than only a set of winning committees. This model is very versatile since with ABC ranking rules one can easily combine the societal evaluation of committees with additional requirements one would like to impose on the structure of the committee. For example, suppose the goal is to select a committee subject to certain diversity constraints (such as an equal number of men and women). In such scenarios, a ranking rule can be applied directly: among the committees that satisfy the diversity constraint, one can simply select the committee that appears highest in the societal ranking.

1.1. Main results

The first main result of this paper is an axiomatic characterization of *ABC scoring rules*, which are a subclass of ABC ranking rules. ABC scoring rules are informally defined as follows: given a real-valued function $f(x, y)$ (the so-called *approval scoring function*), a committee W receives a score of $f(x, y)$ from every voter who approves x candidates in W and who approves y candidates in total; the ABC scoring rule implemented by f ranks committees according to the sum of scores obtained from all voters. We obtain the following characterization.

Theorem 1. *An ABC ranking rule is an ABC scoring rule if and only if it satisfies symmetry, consistency, weak efficiency, and continuity.*

As weak efficiency is satisfied by every sensible multi-winner rule and continuity typically only rules out the use of certain tie-breaking mechanisms (Smith, 1973; Young, 1974a, 1975), Theorem 1 essentially implies that ABC scoring rules correspond to symmetric and consistent ABC ranking rules. Furthermore, we show that the set of axioms used to characterize ABC scoring rules is minimal.

Our second main result is the axiomatic explanation of the differences between three important ABC scoring rules: Proportional Approval Voting (PAV), Approval Chamberlin–Courant (CC), and Multi-Winner Approval Voting (AV). These three well-known rules are prime examples of multi-winner systems following the principle of proportionality, diversity, and individual excellence, respectively. Our results imply that the differences between these three rules can be understood by studying how these rules behave when viewed as *apportionment methods* (Balinski and Young, 1982; Pukelsheim, 2014). Apportionment methods are a well-studied special case of approval-based multi-winner voting, where only party-list profiles are considered. As mentioned before, it is easier to formalize the principles of proportionality, diversity, and individual excellence for these mathematically much simpler profiles:

D’Hondt proportionality defines a way in which parliamentary seats are assigned to parties in a proportional manner. The D’Hondt method (also known as Jefferson method) is one of the most commonly used methods of apportionment in parliamentary elections.

Disjoint diversity states that as many parties as possible should receive one seat and, if necessary, priority is given to stronger parties. Disjoint diversity is implied by apportionment methods such as Huntington–Hill, Dean, or Adams.

Disjoint equality states that if each candidate is approved by at most one voter, then any committee consisting of approved candidates is a winning committee. One can argue that the principle of individual excellence implies disjoint equality: if every candidate is approved only once, then every approved candidate has the same support, their “quality” cannot

Table 1

ABC rules and axioms they satisfy (+) or fail (blank). We use the symbol \oplus instead of + if the axiom is involved in an axiomatic characterization of the rule or class of rules. Classes of rules (ABC scoring rules and Thiele methods) satisfy an axiom if all rules in the class satisfy it; they fail an axiom if one rule in the class fails it.

	Anonymity	Neutrality	Consistency	Weak efficiency	Continuity	D'Hondt prop.	Lower quota	Disjoint equality	Disjoint diversity
ABC scoring rules	\oplus	\oplus	\oplus	\oplus	\oplus				
Thiele methods	+	+	+	+	+				
Multi-Winner Approval Voting (AV)	\oplus	\oplus	\oplus	\oplus	\oplus			\oplus	
Proportional Approval Voting (PAV)	\oplus	\oplus	\oplus	+	\oplus	\oplus	+		
Approval Chamberlin–Courant (CC)	\oplus	\oplus	\oplus	\oplus	\oplus				\oplus
Satisfaction Approval Voting	+	+	+	+	+				

be distinguished, and hence all approved candidates are equally well suited for selection.²

We show that Proportional Approval Voting is the only ABC scoring rule satisfying D'Hondt proportionality, that Approval Chamberlin–Courant is the only ABC scoring rule satisfying disjoint diversity, and that Multi-Winner Approval Voting is the only ABC scoring rule that satisfies disjoint equality. Together with Theorem 1, these results lead to axiomatic characterizations of PAV, CC, and AV within the general class of ABC ranking rules. In particular, our results show that Proportional Approval Voting is essentially the only consistent extension of the D'Hondt method to the more general setting where voters decide on individual candidates rather than on parties. Table 1 summarizes the axiomatic characterizations given in the paper.

We furthermore present two extensions of our main results. First, we show that approval scoring functions that are not “close” to f_{PAV} (all those not contained in the gray area around f_{PAV} in Fig. 1) implement ABC ranking rules that violate a rather weak form of proportionality called lower quota. Second, we show that our characterization of PAV can be generalized to a broader class of ABC scoring rules: given a certain kind of proportionality on party-list profiles, represented as a specific divisor apportionment method, we show that there is a unique symmetric, consistent and continuous ABC ranking rule that guarantees this kind of proportionality.

We postpone the discussion of related literature to Section 6.

2. Preliminaries

We write $[n]$ to denote the set $\{1, \dots, n\}$ and $[i, j]$ to denote $\{i, i + 1, \dots, j\}$ for $i \leq j \in \mathbb{N}$. For a set X , let $\mathcal{P}(X)$ denote the powerset of X , i.e., the set of subsets of X . Further, for each ℓ let $\mathcal{P}_\ell(X)$ denote the set of all size- ℓ subsets of X . A weak order of X is a binary relation that is transitive and complete (all elements of X are comparable), and thus also reflexive. A linear

² In our characterization, disjoint equality could be replaced with a strictly stronger axiom requiring that, in party-list profiles, candidates with a maximum number of approvals must be contained in winning committees. The connection between this stronger axiom and individual excellence is easier to observe. We use disjoint equality in our characterization as it is a weaker assumption and thus strengthens the result.

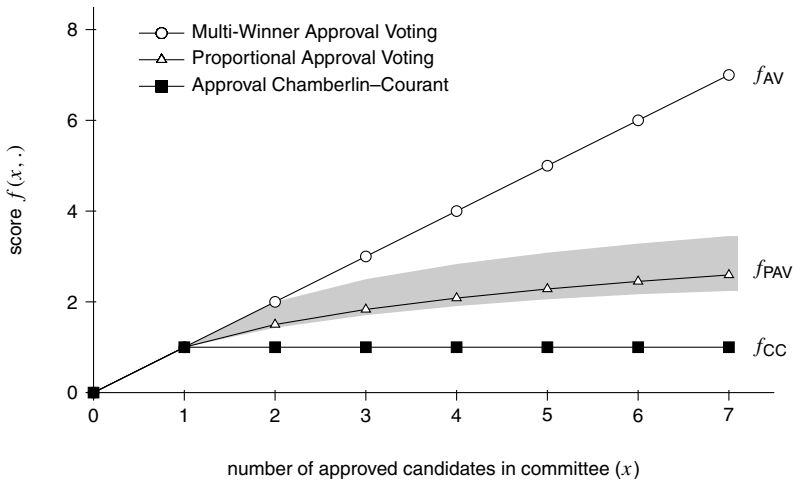


Fig. 1. Different approval scoring functions (see Section 3.1 for definitions) and their corresponding ABC scoring rules. Approval scoring functions outside the gray area fail the lower quota axiom; see Section 5.1 for a formal statement.

order is a weak order that is antisymmetric. We write $\mathcal{W}(X)$ to denote the set of all weak orders of X and $\mathcal{L}(X)$ to denote the set of all linear orders of X .

Approval profiles. Let $C = \{c_1, \dots, c_m\}$ be a set of candidates, and let k denote the desired size of the committee to be formed. We refer to elements of $\mathcal{P}_k(C)$ as *committees*. Throughout the paper, we assume that both k and C (and thus m) are arbitrary but fixed. Furthermore, to avoid trivialities, we assume $1 \leq k < m$.

We identify voters with natural numbers, i.e., \mathbb{N} is the set of all possible voters. Let \mathcal{V} denote the set of all finite subsets of \mathbb{N} . For each finite subset of voters $V = \{v_1, \dots, v_n\} \in \mathcal{V}$, an *approval profile* of V is a function from V to $\mathcal{P}(C)$; we write $A = (A(v_1), \dots, A(v_n))$ as a short-form for this function. For $v \in V$, let $A(v) \subseteq C$ denote the subset of candidates approved by voter v . We write $\mathcal{A}(V)$ to denote the set of all possible approval profiles of V and $\mathcal{A} = \bigcup_{V \in \mathcal{V}} \mathcal{A}(V)$ to be the set of all approval profiles with finite voters sets. (We do not mention C in this notation due to our assumption that C is fixed.) Given a permutation $\sigma : C \rightarrow C$ and $X \subseteq C$, let $\sigma(X) = \{\sigma(c) : c \in X\}$. Further, for an approval profile $A \in \mathcal{A}(V)$, we write $\sigma(A)$ to denote the profile $(\sigma(A(v_1)), \dots, \sigma(A(v_n)))$.

Let $V = \{v_1, \dots, v_s\} \in \mathcal{V}$ and $V' = \{v'_1, \dots, v'_t\} \in \mathcal{V}$. Further, let $A \in \mathcal{A}(V)$ and $A' \in \mathcal{A}(V')$. If V and V' do not intersect, we write $A + A'$ to denote the profile $B \in \mathcal{A}(V \cup V')$ defined as $B = (A(v_1), \dots, A(v_s), A'(v'_1), \dots, A'(v'_t))$. If V and V' intersect, we relabel the voters to $V'' = [1, s + t]$ and define $B \in \mathcal{A}(V'')$ analogously.³ For a positive integer n , we write nA to denote $A + A + \dots + A$, n times.

Approval-based committee ranking rules. An *approval-based committee ranking rule* (ABC ranking rule), $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{W}(\mathcal{P}_k(C))$, maps approval profiles to weak orders over committees. Note that C and k are parameters for ABC ranking rules but—since we assume that C and k are

³ We define $A + A'$ for disjoint V and V' in this way so that we do not implicitly assume anonymity, i.e., that the identity of voters is irrelevant (see Section 3.2). This is necessary so that the consistency axiom (also Section 3.2) is independent of anonymity.

fixed—we omit them to alleviate notation. For an ABC ranking rule \mathcal{F} and an approval profile A , we write $\succeq_{\mathcal{F}(A)}$ to denote the weak order $\mathcal{F}(A)$. For $W_1, W_2 \in \mathcal{P}_k(C)$, we write $W_1 \succ_{\mathcal{F}(A)} W_2$ if $W_1 \succeq_{\mathcal{F}(A)} W_2$ and not $W_2 \succeq_{\mathcal{F}(A)} W_1$, and we write $W_1 \sim_{\mathcal{F}(A)} W_2$ if $W_1 \succeq_{\mathcal{F}(A)} W_2$ and $W_2 \succeq_{\mathcal{F}(A)} W_1$. A committee is a *winning committee* if it is a maximal element with respect to $\succeq_{\mathcal{F}(A)}$.

An ABC ranking rule is *trivial* if for all $A \in \mathcal{A}$ and $W_1, W_2 \in \mathcal{P}_k(C)$ it holds that $W_1 \sim_{\mathcal{F}(A)} W_2$.

Let us now list some important examples of ABC ranking rules.

Thiele Methods (Thiele, 1895). Consider a sequence of positive weights $w = (w_1, w_2, \dots, w_k)$ and define the w -score of a committee W as

$$\sum_{v \in V} \sum_{j=1}^{|W \cap A(v)|} w_j,$$

i.e., if voter v has x approved candidates in W , then W receives a score of $w_1 + w_2 + \dots + w_x$ from v . The w -Thiele method ranks the committees according to their w -scores.

Proportional Approval Voting (PAV). PAV is a Thiele method defined by the weights $w = (1, 1/2, 1/3, \dots)$.

Approval Chamberlin–Courant (CC). The Approval Chamberlin–Courant rule is a Thiele method defined by the weights $w_{CC} = (1, 0, 0, \dots)$. Consequently, CC chooses committees so as to maximize the number of voters who have at least one approved candidate in the winning committee.

Multi-Winner Approval Voting (AV). AV is a Thiele method defined by the weights $w_{AV} = (1, 1, 1, \dots)$. Equivalently, with AV each candidate $c \in C$ obtains one point from each voter who approves c , and the AV-score of a committee W is the total number of points awarded to members of W , i.e., $\sum_{c \in W} |\{v \in V : c \in A(v)\}|$.

3. ABC scoring rules

In this section we define a new class of multi-winner rules, called ABC scoring rules. ABC scoring rules can be viewed as an adaptation of positional scoring rules (Smith, 1973; Young, 1974a) to the world of approval-based multi-winner rules. Furthermore, ABC scoring rules can be viewed as analogous to the class of (multi-winner) committee scoring rules as introduced by Elkind et al. (2017) but defined for approval ballots instead of ranked ballots.

After formally defining ABC scoring rules and introducing some basic axioms, we will present our main technical result: an axiomatic characterization of the class of ABC scoring rules. This result forms the basis for our subsequent axiomatic analysis.

3.1. Defining ABC scoring rules

An *approval scoring function* is a mapping $f : [0, k] \times [0, m] \rightarrow \mathbb{R}$ satisfying $f(x, y) \geq f(x', y)$ for $x \geq x'$. The intuitive meaning is that $f(x, y)$ denotes the score that a committee W obtains from a voter that approves of x members of W and y candidates in total. Let $A \in \mathcal{A}(V)$. We define the score of W in A as

$$sc_f(W, A) = \sum_{v \in V} f(|A(v) \cap W|, |A(v)|). \tag{1}$$

We say that an approval scoring function f implements an ABC ranking rule \mathcal{F} if for every $A \in \mathcal{A}$ and $W_1, W_2 \in \mathcal{P}_k(C)$,

$$sc_f(W_1, A) > sc_f(W_2, A) \quad \text{if and only if} \quad W_1 \succ_{\mathcal{F}(A)} W_2.$$

An ABC ranking rule \mathcal{F} is an ABC scoring rule if there exists an approval scoring function f that implements \mathcal{F} .

As we have seen in the introduction, PAV, CC, and AV are ABC scoring rules and can be implemented by the following approval scoring function:

$$f_{PAV}(x, y) = \sum_{i=1}^x 1/i, \quad f_{CC}(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \geq 1, \end{cases} \quad f_{AV}(x, y) = x.$$

Further, ABC scoring rules include rules such as Constant Threshold Methods (Fishburn and Pekec, 2004) and Satisfaction Approval Voting (Brams and Kilgour, 2014), implemented by

$$f_{CT}(x, y) = \begin{cases} 0 & \text{if } x < t, \\ 1 & \text{if } x \geq t \end{cases} \quad \text{and} \quad f_{SAV}(x, y) = \frac{x}{y}.$$

Note that only Satisfaction Approval Voting is implemented by an approval scoring function depending on y . As can easily be verified, Thiele methods are exactly those ABC scoring rules that can be implemented by an approval scoring function not dependent on y : the approval scoring function f defining w -Thiele is $f(x, y) = w_1 + \dots + w_x$ and, conversely, every approval scoring function $f(x, y)$ that is independent of y can be written as $f(x, y) = \sum_{j=1}^x w_j$ for some sequence of positive weights (w_1, \dots, w_k) .

It is apparent that not the whole domain of an ABC scoring rule is relevant; consider for example $f(2, 1)$ or $f(0, m)$ —these function values will not be used in the score computation of any committee, cf. Equation (1). The following proposition provides a tool for showing that two ABC scoring rules are equivalent. It shows which part of the domain of ABC scoring rules is relevant and that affine transformations yield equivalent rules.

Proposition 1. *Let $D_{m,k} = \{(x, y) \in [0, k] \times [0, m - 1] : x \leq y \wedge k - x \leq m - y\}$ and let f, g be approval scoring functions. If there exist $c \in \mathbb{R}$ and $d : [m] \rightarrow \mathbb{R}$ such that $f(x, y) = c \cdot g(x, y) + d(y)$ for all $x, y \in D_{m,k}$ then f, g implement the same ABC scoring rule, i.e., for all approval profiles $A \in \mathcal{A}(V)$ and committees $W_1, W_2 \in \mathcal{P}_k(C)$ it holds that $sc_f(W_1, A) > sc_f(W_2, A)$ if and only if $sc_g(W_1, A) > sc_g(W_2, A)$.*

3.2. Basic axioms

In this section, we discuss formal definitions of the axioms used for our main characterization result (Theorem 1). All axioms are natural and straightforward adaptations of the respective properties of single-winner election rules. Similar axioms have also been considered in the context of multi-winner rules for the model where voters express their preferences by ranking the candidates (Elkind et al., 2017; Skowron et al., 2019).

Anonymity and neutrality enforce perhaps the most basic fairness requirements for voting rules. Anonymity is a property which requires that all voters are treated equally, i.e., the result of an election does not depend on particular names of voters but only on votes that have been cast. In other words, under anonymous ABC ranking rules, each voter has the same voting power. Neutrality is similar, but enforces equal treatment of candidates rather than of voters.

Anonymity. An ABC ranking rule \mathcal{F} is *anonymous* if for $V, V' \in \mathcal{V}$ such that $|V| = |V'|$, for each bijection $\rho : V \rightarrow V'$, and for $A \in \mathcal{A}(V)$ and $A' \in \mathcal{A}(V')$ such that $A(v) = A'(\rho(v))$ for each $v \in V$, it holds that $\mathcal{F}(A) = \mathcal{F}(A')$.

Neutrality. An ABC ranking rule \mathcal{F} is *neutral* if for each bijection $\sigma : C \rightarrow C$ and $A, A' \in \mathcal{A}$ with $\sigma(A) = A'$ it holds for $W_1, W_2 \in \mathcal{P}_k(C)$ that $W_1 \succeq_{\mathcal{F}(A)} W_2$ if and only if $\sigma(W_1) \succeq_{\mathcal{F}(A')} \sigma(W_2)$.

Due to their analogous structure and similar interpretations, anonymity and neutrality are very often considered together, and jointly referred to as symmetry.

Symmetry. An ABC ranking rule is *symmetric* if it is anonymous and neutral.

Consistency was first introduced in the context of single-winner rules by Smith (1973) and then adapted by Young (1974a). In the world of single-winner rules, consistency is often considered to be *the* axiom that characterizes positional scoring rules. Similarly, consistency played a crucial role in the recent characterization of committee scoring rules (Skowron et al., 2019), which can be considered the equivalent of positional scoring rules in the multi-winner setting. Consistency is also the main ingredient for our axiomatic characterization of ABC scoring rules.

Consistency. An ABC ranking rule \mathcal{F} is *consistent* if for disjoint $V, V' \in \mathcal{V}$, $A \in \mathcal{A}(V)$, $A' \in \mathcal{A}(V')$, and $W_1, W_2 \in \mathcal{P}_k(C)$, it holds that

- (i) if $W_1 \succ_{\mathcal{F}(A)} W_2$ and $W_1 \succeq_{\mathcal{F}(A')} W_2$, then $W_1 \succ_{\mathcal{F}(A+A')} W_2$, and
- (ii) if $W_1 \succeq_{\mathcal{F}(A)} W_2$ and $W_1 \succ_{\mathcal{F}(A')} W_2$, then $W_1 \succeq_{\mathcal{F}(A+A')} W_2$.

Next, we describe a weak efficiency axiom, which captures the intuition that candidates approved by no one are undesirable.

Weak efficiency. An ABC ranking rule \mathcal{F} satisfies *weak efficiency* if for each $W_1, W_2 \in \mathcal{P}_k(C)$ and $A \in \mathcal{A}$ where no voter approves a candidate in $W_2 \setminus W_1$, it holds that $W_1 \succeq_{\mathcal{F}(A)} W_2$.

For $k = 1$, i.e., in the single-winner setting, we see that weak efficiency reduces to the following statement: if no voter approves a candidate d , then any other candidate is at least as preferable as d .

The final axiom, continuity (Young, 1974a, 1975) (also known in the literature as the Archimedean property (Smith, 1973) or the Overwhelming Majority axiom (Myerson, 1995)), describes the influence of large majorities in the process of making a decision. Continuity enforces that a large enough group of voters is able to force the election of their most preferred

committee. Continuity is pivotal in Young’s characterizations of positional scoring rules (Young, 1974a, 1975) as it excludes specific tie-breaking mechanisms.⁴

Continuity. An ABC ranking rule \mathcal{F} satisfies *continuity* if for each $W_1, W_2 \in \mathcal{P}_k(C)$ and $A, A' \in \mathcal{A}$ where $W_1 \succ_{\mathcal{F}(A')} W_2$ there exists a positive integer n such that $W_1 \succ_{\mathcal{F}(A+nA')} W_2$.

3.3. A characterization of ABC scoring rules

The following axiomatic characterization of the generic class of ABC scoring rules is a powerful tool that forms the basis for further characterizations of specific ABC scoring rules. This result resembles Smith’s and Young’s characterization of positional scoring rules (Young, 1974a; Smith, 1973) as the only social welfare functions satisfying symmetry, consistency, and continuity. Our characterization additionally requires weak efficiency, which stems from the condition that an approval scoring function $f(x, y)$ must be weakly increasing in x . If a similar condition was imposed on positional scoring rules (i.e., that positional scores are weakly decreasing), an axiom analogous to weak efficiency would be required for a characterization as well.

Theorem 1. *An ABC ranking rule is an ABC scoring rule if and only if it satisfies symmetry, consistency, weak efficiency, and continuity.*

It is easy to verify that ABC scoring rules satisfy symmetry, consistency, weak efficiency, and continuity; all this follows immediately from the definitions in Section 3.1, in particular the summation in Equation (1). For example, consistency is an immediate consequence of the fact that $sc_f(W, A + A') = sc_f(W, A) + sc_f(W, A')$. In Appendix A we provide the proof of the other implication. This proof relies on the axiomatic characterization of committee scoring rules (Skowron et al., 2019). Committee scoring rules are multi-winner voting rules that accept preferences in the form of linear orders as input and output a ranking of committees (a definition can be found in Appendix A). The main difference to ABC ranking rules is thus the type of preferences. It is important to note that the characterization of Skowron et al. (2019) only holds for linear orders and not for weak orders, hence ABC ranking rules are not covered by their result. On the contrary, it requires substantial work to prove the exact relation between these two classes so that results can be transferred from one class to the other.

The set of axioms used in Theorem 1 is minimal, i.e., any subset of axioms is not sufficient for the characterization statement to hold (see Appendix A.4).

4. Proportional and disproportional ABC scoring rules

In this section we consider axioms describing winning committees in party-list profiles and capture a specific variant of proportionality, individual excellence, or diversity. In party-list profiles, voters and candidates are grouped into clusters, which can be intuitively viewed as political parties. We will show that axioms for party-list profiles are sufficient to characterize certain ABC

⁴ In Young’s characterization, continuity excludes *composite* positional scoring rules, where one or more additional positional scoring rules are evaluated in case of ties (same score). Other tie-breaking mechanisms are already excluded by consistency and symmetry.

Table 2
Example for the D'Hondt method.

	N_1	N_2	N_3	N_4
$ N_i /1$	9	21	28	42
$ N_i /2$	4.5	10.5	14	21
$ N_i /3$	3	7	13	14
$ N_i /4$	2.25	5.25	7	10.5
$ N_i /5$	1.8	4.2	5.6	8.4

scoring rules: PAV, AV, and CC. Using the axiomatic characterization of ABC scoring rules (Theorem 1), we obtain full axiomatic characterizations of these three rules.

Definition 1. An approval profile is a *party-list profile with p parties* if the set of voters can be partitioned into pairwise disjoint sets N_1, N_2, \dots, N_p and the set of candidates can be partitioned into pairwise disjoint sets C_1, C_2, \dots, C_p such that, for each $i \in [p]$, every voter v in N_i approves C_i and no other candidates, i.e., $A(v) = C_i$.

In other words, an approval profile is a party-list profile if for any two voters their approval sets are either the same or disjoint.

4.1. D'Hondt proportionality

In party-list profiles, we intuitively expect a proportional committee to contain as many candidates from a party as is proportional to the number of this party's supporters. There are numerous ways in which this concept can be formalized—different notions of proportionality are expressed through different methods of apportionment (Balinski and Young, 1982; Pukelsheim, 2014). In this section we consider one of the best known, and most commonly used concept of proportionality: *D'Hondt proportionality*. The D'Hondt method is an apportionment method that works in k steps. It starts with an empty committee $W = \emptyset$ and in each step it selects a candidate from that set (party) C_i with a maximal value of $\frac{|N_i|}{|W \cap C_i| + 1}$; the selected candidate is added to W .

Example 1. Consider an election with four groups of voters, N_1, N_2, N_3 , and N_4 with cardinalities equal to 9, 21, 28, and 42, respectively. Further, there are four groups of candidates $C_1 = \{c_1, \dots, c_{10}\}$, $C_2 = \{c_{11}, \dots, c_{20}\}$, $C_3 = \{c_{21}, \dots, c_{30}\}$, and $C_4 = \{c_{31}, \dots, c_{40}\}$. Voters in a group N_i approve exactly the candidates in C_i . Assume $k = 10$ and consider Table 2, which illustrates the ratios used in the D'Hondt method for determining which candidate should be selected. The 10 largest values (in bold) correspond to selected candidates.

Thus, the D'Hondt method first selects a candidate from C_4 , next a candidate from C_3 , next from C_2 or C_4 (their ratios in the third step are equal), etc. Eventually, in the selected committee there will be one candidate from C_1 , two candidates from C_2 , three from C_3 , and four from C_4 .

An important difference between the apportionment setting and our setting is that we do not necessarily assume an unrestricted number of candidates for each party. As a consequence, a party might deserve additional candidates but this is impossible to fulfill. Taking this restriction into account, we see that if the D'Hondt method picks a candidate from C_i and adds it to W ,

then, for all j , either $\frac{|N_i|}{|W \cap C_i|} \geq \frac{|N_j|}{|W \cap C_j|+1}$ or $C_j \subseteq W$, i.e., all candidates from party j are already in the committee. Note that if $C_j \setminus W \neq \emptyset$ and $\frac{|N_i|}{|W \cap C_i|} < \frac{|N_j|}{|W \cap C_j|+1}$, then the D'Hondt method in the previous step would rather select a candidate from C_j than from C_i . These observations allow us to give a precise definition of D'Hondt proportional committees.

Definition 2. Let A be a party-list profile with p parties. A committee $W \in \mathcal{P}_k(C)$ is *D'Hondt proportional* for A if for all $i, j \in [p]$, if $W \cap C_i \neq \emptyset$, then either $C_j \subseteq W$ or $\frac{|N_i|}{|W \cap C_i|} \geq \frac{|N_j|}{|W \cap C_j|+1}$.

For the following axiom, recall that a winning committee is a maximal element in social ranking of committees, i.e., with respect to $\succeq_{\mathcal{F}(A)}$.

D'Hondt proportionality. An ABC ranking rule satisfies *D'Hondt proportionality* if for each party-list profile $A \in \mathcal{A}$, $W \in \mathcal{P}_k(C)$ is a winning committee if and only if W is D'Hondt proportional for A .

Note that this axiom is weak in the sense that it only describes the expected behavior of an ABC ranking rule on party-list profiles. As we will see, however, it is sufficient to obtain an axiomatic characterization of PAV in the more general framework of ABC ranking rules.

Theorem 2. *Proportional Approval Voting is the only ABC scoring rule that satisfies D'Hondt proportionality.*

When we combine Theorem 2 with Theorem 1, we obtain a full axiomatic characterization of Proportional Approval Voting within the class of ABC ranking rules: PAV is the only ABC ranking rule that satisfies symmetry, consistency, continuity, and D'Hondt proportionality. Note the absence of weak efficiency in the set of axioms that characterize PAV, since weak efficiency is implied by the other axioms (cf. Lemma 1 in Section 5.2). In Section 5.2, we will present a generalization of Theorem 2 that applies to apportionment methods other than D'Hondt.

Finally, we note that Theorem 2 shows that within the class of ABC scoring rules a weak proportionality axiom such as D'Hondt proportionality already suffices to imply much stronger proportionality guarantees: PAV satisfies axioms such as extended justified representation (Aziz et al., 2017) and proportional justified representation (Sánchez-Fernández et al., 2017).

4.2. Disjoint diversity

The disjoint diversity axiom is strongly related to the diversity principle, as it states that there exists a winning committee in which the k strongest parties receive at least one seat. In other words, every party has to receive one seat before one party receives a second seat.

Disjoint diversity. An ABC ranking rule \mathcal{F} satisfies *disjoint diversity* if for every party-list profile $A \in \mathcal{A}$ with p parties and $|N_1| \geq |N_2| \geq \dots \geq |N_p|$, there exists a winning committee W with $W \cap C_i \neq \emptyset$ for all $i \in \{1, \dots, \min(p, k)\}$.

Note that disjoint diversity is a slightly weaker axiom in comparison to D'Hondt proportionality since it does not characterize all winning committees for party-list profiles—it only requires

the existence of one specific winning committee and does not even fully specify this committee. As a consequence, there are several apportionment methods in the literature that imply disjoint diversity: the Adams method, the Dean method, and the Huntington–Hill method all require that every party receives one seat before a party can obtain a second seat (Balinski and Young, 1982). Thus, it may come as a surprise that disjoint diversity nevertheless characterizes a single ABC scoring rule.

Theorem 3. *The Approval Chamberlin–Courant rule is the only non-trivial ABC scoring rule that satisfies disjoint diversity.*

Observe that CC does not extend the aforementioned apportionment methods because of the simple fact that it is not at all proportional. We can thus conclude that these apportionment methods do not have a counterpart in the class of ABC scoring rules. However, if we allow a tie-breaking mechanism, we find analogues. For example, the Adams method is a divisor method similar to D’Hondt but based on the divisor sequence $(0, 1, 2, \dots)$. As vote counts are first divided by 0 (defined as an arbitrarily large number), each party is guaranteed to receive one seat. The Adams method can be extended to a ABC ranking rule: it is the Chamberlin–Courant rule with the $(w_1, 1, 1/2, 1/3, \dots)$ -Thiele method used to break ties between committees with the same CC score (w_1 is an arbitrary number).

Finally, we obtain as a corollary that CC is characterized as the only non-trivial ABC ranking rule that satisfies symmetry, consistency, weak efficiency, continuity, and disjoint diversity.

4.3. Disjoint equality

In some scenarios, we might want a multi-winner rule to be neither proportional nor diverse. For example, if our goal is to select a set of finalists in a contest based on a set of recommendations coming from judges or reviewers (a scenario that is often referred to as a shortlisting), candidates can be assessed independently and there is no need for proportionality. For instance, if our goal is to select 5 finalists in a contest, and if four reviewers support candidates c_1, \dots, c_5 and one reviewer supports candidates c_6, \dots, c_{10} then it is very likely that we would prefer to select candidates c_1, \dots, c_5 as the finalists—in contrast to what, e.g., D’Hondt proportionality or disjoint diversity suggest.

Disjoint equality is a property which might be viewed as a certain type of disproportionality. Intuitively, it requires that each approval of a candidate carries the same power: a candidate approved by a voter v receives a certain level of support from v which does not depend on what other candidates v approve or disapprove of; in particular it does not depend on whether there are other members of a winning committee which are approved by v . Disjoint equality was first proposed by Fishburn (1978) and then used by Sertel (1988) as one of the distinctive axioms characterizing single-winner Approval Voting. The following axiom is its natural extension to the multi-winner setting.

Disjoint equality. An ABC ranking rule \mathcal{F} satisfies *disjoint equality* if for every voters set $V \in \mathcal{V}$, profile $A \in \mathcal{A}(V)$ with $|\bigcup_{v \in V} A(v)| \geq k$ and where each candidate is approved at most once, the following holds: $W \in \mathcal{P}_k(C)$ is a winning committee if and only if $W \subseteq \bigcup_{v \in V} A(v)$.

In other words, disjoint equality asserts that—in a profile consisting of disjoint approval ballots—every committee wins that consists of approved candidates. Note that disjoint equality applies to an even more restricted form of party-list profiles.

Theorem 4. *Multi-Winner Approval Voting is the only ABC scoring rule that satisfies disjoint equality.*

Theorem 4 together with Theorem 1 yields an axiomatic characterization: AV is the only ABC ranking rule that satisfies symmetry, consistency, weak efficiency, continuity, and disjoint equality.

5. Extensions

In this section we discuss two extensions of our main results. First, we define a weaker form of D’Hondt proportionality, called lower quota, and we show that ABC rules that satisfy lower quota must resemble PAV. Second, we extend our axiomatic characterizations of PAV and show a more general result that applies to a whole spectrum of different forms of proportionality.

5.1. Lower quota

D’Hondt proportionality determines for every party-list profile an apportionment of candidates to parties. One may wonder if this definition of proportionality can be further weakened and still allow a characterization of PAV. For example, the D’Hondt method is the only divisor method satisfying the lower quota axiom (Balinski and Young, 1982): intuitively, it states that a party that receives an α proportion of votes should receive at least $\lfloor \alpha \cdot k \rfloor$ of the k available seats. In the following we will show that this weaker axiom is not sufficient, but it characterizes ABC scoring rules that are at least similar to PAV. Let us first define lower quota for ABC ranking rules:

Lower Quota. An ABC ranking rule satisfies *lower quota* if for each party-list profile A with p parties, and winning committee $W \in \mathcal{P}_k(C)$ it holds for all $i \in \{1, \dots, p\}$ that $|W \cap C_i| \geq \lfloor \frac{k|N_i|}{|V|} \rfloor$ or $|C_i| < \lfloor \frac{k|N_i|}{|V|} \rfloor$.

First, let us observe that there exist ABC scoring rules—other than PAV—which satisfy lower quota.

Example 2. Let $m = 3$ and $k = 2$. Let us consider an ABC scoring rule defined by the approval scoring function $f(0, y) = 0$, and $f(1, y) = 1$ and $f(2, y) = 1.1$. This rule satisfies lower quota: Let A be a party-list profile for $m = 3$ with $p \leq 3$ disjoint groups of voters N_1, N_2, \dots, N_p and with their corresponding approval sets being C_1, \dots, C_p . For the sake of contradiction, let us assume that there exists a winning committee W such that for some $i \in [p]$ we have $|C_i| \geq \lfloor 2 \cdot \frac{|N_i|}{|V|} \rfloor$ and $|W \cap C_i| < \lfloor 2 \cdot \frac{|N_i|}{|V|} \rfloor$. If $N_i = V$, then this means that a candidate who is not approved by any voter is contained in W , which contradicts the definition of our rule and the fact that there exist two candidates approved by some voters (since $|N_i| = |V|$, we get that $|C_i| \geq 2$). If $|N_i| < |V|$, then $\lfloor 2 \cdot \frac{|N_i|}{|V|} \rfloor$ can either be 0 or 1. Since $|W \cap C_i| < \lfloor 2 \cdot \frac{|N_i|}{|V|} \rfloor$, we conclude that

$\lfloor 2 \cdot \frac{|N_i|}{|V|} \rfloor = 1$ and $|W \cap C_i| = 0$. Consequently $|N_i| \geq \frac{|V|}{2}$; even if all the remaining voters from $V \setminus N_i$ approved the two members of the winning committee W it is more beneficial, according to our rule, to drop one such candidate from W and to add a candidate from C_i . Indeed, it is easy to verify that such a committee would have a higher score. This shows that our rule indeed satisfies lower quota.

The following shows that ABC scoring rules satisfying lower quota must resemble PAV.

Proposition 2. Fix $x, y \in \mathbb{N}$ and let $m \geq y + k - x + 1$. Let \mathcal{F} be an ABC scoring rule satisfying lower quota, and let f be an approval scoring function implementing \mathcal{F} . It holds that:

$$f(x - 1, y) + \frac{1}{x} \cdot f(1, 1) \cdot \frac{k - x}{k - x + 1} \leq f(x, y) \leq f(x - 1, y) + \frac{1}{x - 1} \cdot f(1, 1).$$

Note that $\lim_{k \rightarrow \infty} \frac{k - x}{k - x + 1} = 1$, so Proposition 2 asserts that—for large k —the value of $f(x, y)$ is roughly between $f(x - 1, y) + \frac{1}{x} \cdot f(1, 1)$ and $f(x - 1, y) + \frac{1}{x - 1} \cdot f(1, 1)$. Recall that for PAV we have that $f(x, y) = f(x - 1, y) + \frac{1}{x} \cdot f(1, 1)$ and hence Proposition 2 indeed implies that an ABC scoring rule satisfying lower quota must be defined by an approval scoring function similar to PAV.

For a visualization of this result, we recall Fig. 1 in the introduction of this paper. The gray area displays the lower and upper bound obtained from Proposition 2; to compute a lower bound we used $k = 8$.

5.2. Extension to other forms of proportionality

In this section we formulate an axiom that generalizes D’Hondt proportionality. Given a sequence $\mathbf{d} = (d_1, d_2, \dots)$, the \mathbf{d} -proportionality requires that a multi-winner rule must behave on party-list profiles as a divisor apportionment method based on the sequence of divisors d . Thus, for the sequence $d_{\text{D’Hondt}} = (1, 2, 3, \dots)$, $d_{\text{D’Hondt}}$ -proportionality is equivalent to D’Hondt proportionality. Notably, this definition applies to other known apportionment divisor methods, such as the *Sainte-Laguë* (Webster) method—the divisor method based on the sequence $d_{\text{SL}} = (1, 3, 5, \dots)$. It also applies to non-linear forms of proportionality—for example, the sequence of divisors $d_{\text{Penrose}} = (1, 4, 9, \dots)$ implements the idea of square-root proportionality as devised by Penrose (1946), where a party should get a number of seats proportional to the square root of the number of supporters. In the following we use the convention that $\frac{x}{\infty} = 0$ for integers x .

Definition 3. Let A be a non-decreasing party-list profile with p parties, and $\mathbf{d} = (d_i)_{i \in \mathbb{N}}$, where $d_i \in \mathbb{N} \cup \{\infty\}$ for each $i \in \mathbb{N}$. A committee $W \in \mathcal{P}_k(C)$ is \mathbf{d} -proportional for A if for all $i, j \in [p]$ one of the following conditions holds: (i) $C_j \subseteq W$, or (ii) $W \cap C_i = \emptyset$, or (iii) $\frac{|N_i|}{d_{|W \cap C_i|}} \geq \frac{|N_j|}{d_{|W \cap C_{j+1}|}}$.

d-proportionality. Let $\mathbf{d} = (d_i)_{i \in \mathbb{N}}$ be a sequence of values from $\mathbb{N} \cup \{\infty\}$. An ABC ranking rule satisfies \mathbf{d} -proportionality if for each party-list profile $A \in \mathcal{A}(V)$, $W \in \mathcal{P}_k(C)$ is a winning committee if and only if W is \mathbf{d} -proportional for A .

Theorem 5. Let $\mathbf{d} = (d_1, d_2, \dots)$ be a non-decreasing sequence of values from $\mathbb{N} \cup \{\infty\}$ and let $w = (1/d_1, 1/d_2, \dots)$. The w -Thiele method is the only ABC scoring rule that satisfies \mathbf{d} -proportionality.

Theorem 5 contains the characterization of PAV via D'Hondt proportionality as a special case. It also gives a characterization of CC as the only ABC scoring rule that is $(1, \infty, \infty, \dots)$ -proportional. Note that this characterization is slightly weaker than the one via disjoint diversity (Theorem 3), since $(1, \infty, \infty, \dots)$ -proportionality specifies the behavior of the rule on all party-list profiles. Furthermore, we can use Theorem 5 to obtain axiomatic characterizations within the class of ABC ranking rules.

Lemma 1. Let $d = (d_1, d_2, \dots)$ be a non-decreasing sequence of values from \mathbb{N} . An ABC ranking rule that satisfies neutrality, consistency, and \mathbf{d} -proportionality also satisfies weak efficiency.

By combining Theorem 1, Theorem 5, and Lemma 1, we obtain the desired characterization.

Corollary 1. Let $\mathbf{d} = (d_1, d_2, \dots)$ be a non-decreasing sequence of values from $\mathbb{N} \cup \{\infty\}$ and let $w = (1/d_1, 1/d_2, \dots)$. The w -Thiele method is the only ABC ranking rule that satisfies symmetry, consistency, continuity, weak efficiency, and \mathbf{d} -proportionality. If the values from the sequence \mathbf{d} do not contain ∞ , then we do not require weak efficiency to characterize the corresponding w -Thiele method.

6. Related literature

We briefly review literature that is helpful to place our paper in a larger research context. *Multi-winner voting rules* are central to political elections and originate from this context (cf. the books of Farrell 2011, and Renwick and Pilet 2016). In recent years, however, there has been an emerging interest in multi-winner elections from the computer science community. In this context, multi-winner election rules have been analyzed and applied in a variety of scenarios: personalized recommendation and advertisement (Lu and Boutilier, 2011, 2015), group recommendation (Skowron et al., 2016; Chakraborty et al., 2019), diversifying search results (Skowron et al., 2017), improving genetic algorithms (Faliszewski et al., 2016), and the broad class of facility location problems (Farahani and Hekmatfar, 2009; Skowron et al., 2016). In all these settings, multi-winner voting either appears as a core problem itself or can help to improve mechanisms and algorithms. For an overview of this literature, we refer the reader to surveys by Faliszewski et al. (2017) and Lackner and Skowron (2020).

The most important axiomatic concept in our study is *consistency*. Smith (1973) and Young (1974a) independently introduced this axiom and characterized the class of positional scoring rules as the only social welfare functions that satisfy symmetry, consistency, and continuity. Subsequently, Young (1975) proved an analogous result for social choice functions, i.e., rules that return the set of winning candidates. Further, Myerson (1995) and Pivato (2013) characterized positional scoring rules with the same set of axioms but without imposing any restriction on the input of rules, i.e., ballots are not restricted to be a particular type of order. Extensive studies led to further, more specific, characterizations of consistent voting rules (Chebotarev and Shamis,

1998; Merlin, 2003). Consistency is an important concept also in probabilistic social choice: Brandl et al. (2016) characterize Fishburn's rule of maximal lotteries via two consistency axioms.

In contrast to single-winner voting which is largely well-understood and characterized, *axiomatic studies of multi-winner rules* are considerably fewer in number and are mostly studied in the model where the voters express their preferences by ranking the candidates. Debord (1992) characterized the k -Borda rule using similar axioms as Young (1974b). Felsenthal and Maoz (1992) and Elkind et al. (2017) formulated a number of axiomatic properties of multi-winner rules, and analyzed which rules satisfy these axioms. The axiomatic characterization of the class of committee scoring rules by Skowron et al. (2019) plays a major role in the proof of Theorem 1 (details in Appendix A). Faliszewski et al. (2018, 2019) further studied the internal structure of committee scoring rules and characterized several multi-winner rules within this class. A major topic in this field is proportional representation and several notions have been proposed (Dummett, 1984; Aziz et al., 2017; Sánchez-Fernández et al., 2017; Peters and Skowron, 2020); a detailed summary can be found in the survey of Lackner and Skowron (2020). Finally, recent work has demonstrated that proportionality is incompatible with various forms of strategyproofness (Peters, 2018; Lackner and Skowron, 2018b; Kluiving et al., 2020).

7. Conclusions

In this paper we analyzed a variety of different rules which all satisfy four common properties: symmetry, consistency, continuity, and weak efficiency. We identified the class of rules that is uniquely defined by these four properties: ABC scoring rules—to the best of our knowledge, this class has not been studied previously. The class of ABC scoring rules is remarkably broad. It contains several classic ABC ranking rules, such as Proportional Approval Voting (PAV), Approval Chamberlin–Courant (CC), and Multi-Winner Approval Voting (AV). The class of ABC scoring rules contains the class of Thiele methods, which itself is versatile—Thiele methods are those ABC scoring rules that can be defined by approval scoring functions which do not depend on the parameter y (for example, PAV, CC, and AV are Thiele methods). In addition, the class of ABC scoring rules contains all dissatisfaction counting rules, whose defining approval scoring functions depend only on the difference $y - x$ (Lackner and Skowron, 2018b). (Intuitively, according to dissatisfaction counting rules, each voter cares about minimizing the number of approved but not elected candidates. AV is only one rule which belongs to the classes of Thiele methods and dissatisfaction counting rules.) Yet, ABC scoring rules give an extra degree of freedom, compared to Thiele methods and dissatisfaction counting rules, as they also include other interesting voting methods such as Satisfaction Approval Voting, which is defined by the following approval scoring function: $f_{SAV}(x, y) = x/y$. Satisfaction Approval Voting is quite different from the three rules mentioned above. It uses a very specific interpretation of voters' approval ballots—it assumes each voter is initially given one point, which she can split equally among a set of candidates of her choice; such an interpretation cannot be achieved within the class of Thiele methods or dissatisfaction counting rules.

Our main result, Theorem 1, allows obtaining further, more specific axiomatic characterizations. In particular, as our second main result, we provided axiomatic characterizations of three specific ABC scoring rules: Proportional Approval Voting, Approval Chamberlin–Courant and Multi-Winner Approval Voting. These characterizations are obtained by axioms that describe desirable outcomes for certain simple profiles, so-called party-list profiles. This is a fruitful approach as it is much easier to formally define concepts such as proportionality or diversity on these simple profiles. In such profiles it is also easy to formulate properties which quantify

trade-offs between individual efficiency, proportionality, and diversity. Furthermore, the simpler domain of party-list profiles is sufficient to explain the difference between the rules: PAV, AV, and CC can be obtained by extending three different principles defined for party-list profiles to the more general domain by additionally imposing the same set of axioms. Therefore, their defining differences can be found in party-list profiles.

Our results are general and extend to other concepts definable on party-list profiles, e.g., Sainte-Laguë (Pukelsheim, 2014; Balinski and Young, 1982) or square-root proportionality (Penrose, 1946; Slomczyński and Życzkowski, 2006). Square-root proportionality follows the *degressive proportionality* principle (Koriyama et al., 2013), which suggests that smaller populations should be allocated more representatives than linear proportionality would require. This can be achieved by using a flatter approval scoring function than f_{PAV} and by that we obtain rules which increasingly promote diversity within the committee. An extreme example is the Approval Chamberlin–Courant rule, where the diversity within a winning committee is strongly favored over proportionality. On the other hand, using steeper approval scoring functions results in rules with a utilitarian focus, i.e., rules that tend to select candidates with the high total support from voters, and ignoring issues of proportional representation. Multi-Winner Approval Voting is an extreme example of a rule which does not guarantee virtually any level of proportionality.

Further, Theorem 1 allows to obtain other interesting characterizations, which are not in the main focus of this paper. For example, by adding independence of irrelevant alternatives to the set of axioms listed in Theorem 1 we obtain a characterization of the class of Thiele methods; if we add a variant of monotonicity, we obtain a characterization of the class of dissatisfaction counting rules (Lackner and Skowron, 2018b). Finally, including a variant of strategyproofness yields a characterization of Multi-Winner Approval Voting (Lackner and Skowron, 2018b).

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Appendix A. Characterization of ABC scoring rules

In this section we prove the main technical result of this paper:

Theorem 1. *An ABC ranking rule is an ABC scoring rule if and only if it satisfies symmetry, consistency, weak efficiency, and continuity.*

The following definitions and notation will prove useful:

- For each $\ell \in [0, m]$, we say that an approval profile A is ℓ -regular if each voter in A approves of exactly ℓ candidates.
- We say that A is ℓ -bounded if each voter in A approves of at most ℓ candidates.
- We write $set(A)$ to denote $\{A(v) : v \in V\}$ and by that ignore multiplicities of votes.
- Sometimes we associate an approval set $S \subseteq C$ with the single-voter profile $A \in \mathcal{A}(\{1\})$ such that $A(1) = S$; in such a case we write $\mathcal{F}(S)$ as a short form of $\mathcal{F}(A)$ for appropriately defined A .

Committee scoring rules Before we start describing our construction, let us recall the definition of committee scoring rules (Skowron et al., 2019), a concept that will play an instrumental role in our further discussion. Linear order-based committee (LOC) ranking rules, in contrast to ABC ranking rules, assume that voters’ preferences are given as linear orders over the set of candidates. For a finite set of voters $V = \{v_1, \dots, v_n\} \in \mathcal{V}$, a *profile of linear orders over V* , $P = (P(v_1), \dots, P(v_n))$, is an n -tuple of linear orders over C indexed by the elements of V , i.e., for all $v \in V$ we have $P(v) \in \mathcal{L}(C)$. A *linear order-based committee ranking rule (LOC ranking rule)* is a function that maps profiles of linear orders to $\mathcal{W}(\mathcal{P}_k(C))$, the set of weak orders over committees.

Let P be a profile of linear orders over V . For a vote v and a candidate a , by $pos_v(a, P)$ we denote the position of a in $P(v)$ (the top-ranked candidate has position 1 and the bottom-ranked candidate has position m). For a vote $v \in V$ and a committee $W \in \mathcal{P}_k(C)$, we write $pos_v(W, P)$ to denote the set of positions of all members of W in ranking $P(v)$, i.e., $pos_v(W, P) = \{pos_v(a, P) : a \in W\}$. A *committee scoring function* is a mapping $g : \mathcal{P}_k([m]) \rightarrow \mathbb{R}$ that for each possible position that a committee can occupy in a ranking (there are $\binom{m}{k}$ of all possible positions), assigns a score. Intuitively, for each $I \in \mathcal{P}_k([m])$ value $g(I)$ can be viewed as the score assigned by a voter v to the committee whose members stand in v ’s ranking on positions from set I . Additionally, a committee scoring function $g(I)$ is required to satisfy weak dominance, which is defined as follows. Let $I, J \in \mathcal{P}_k([m])$ such that $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$, and that $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. We say that I dominates J if for each $t \in [k]$ we have $i_t \leq j_t$. Weak dominance holds if I dominating J implies that $g(I) \geq g(J)$.

For a profile of linear orders P over C and a committee $W \in \mathcal{P}_k(C)$, we write $sc_f(W, P)$ to denote the total score that the voters from V assign to committee W . Formally, we have that $sc_g(W, P) = \sum_{v \in V} g(pos_v(W, P))$. An LOC ranking rule \mathcal{G} is an *LOC scoring rule* if there exists a committee scoring function g such that for each $W_1, W_2 \in \mathcal{P}_k(C)$ and profile of linear orders P over V , we have that W_1 is strictly preferred to W_2 with respect to the weak order $\mathcal{G}(P)$ if and only if $sc_g(W_1, P) > sc_g(W_2, P)$.

The axioms from Section 3.2 can be naturally formulated for LOC ranking rules. We will use these formulations of the axioms in the proof of Lemma 3. For the sake of readability we do not recall their definitions here, but rather in the proof, where they are used.

Overview of the proof of Theorem 1 As mentioned before, it is easy to see that ABC scoring rules satisfy symmetry, consistency, weak efficiency, and continuity. The proof of the other direction consists of several steps. Let \mathcal{F} be an ABC ranking rule satisfying symmetry, consistency, weak efficiency, and continuity.

We start in Section A.1 by proving that weak efficiency in conjunction with the other axioms implies a stronger efficiency axiom, which proves useful in the subsequent constructions. In Section A.2, we prove that the characterization theorem holds for the very restricted class of ℓ -regular profiles, i.e., profiles where every voter approves exactly ℓ candidates. To this end,

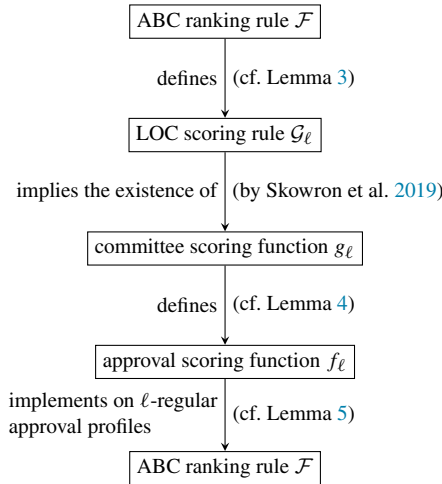


Fig. A.2. A diagram illustrating the reasoning used in Section A.2 to prove that in ℓ -regular approval profiles, \mathcal{F} is an ABC scoring rule.

for each ℓ , we consider the LOC ranking rule \mathcal{G}_ℓ that converts each voter’s linear order into the approval ballot consisting of her top ℓ candidates and then applies \mathcal{F} . We then show that the LOC ranking rule \mathcal{G}_ℓ satisfies equivalent axioms to symmetry, consistency, weak efficiency, and continuity. This allows us to apply a theorem by Skowron et al. (2019), who proved that LOC ranking rules satisfying these axioms are in fact LOC scoring rules. Thus, there exists a corresponding committee scoring function g_ℓ , which in turn defines an approval scoring function f_ℓ . As a last step, we show that f_ℓ implements \mathcal{F} on ℓ -regular approval profiles and thus prove that Theorem 1 holds restricted to ℓ -regular approval profiles.

In Section A.3, we extend this restricted result to arbitrary approval profiles. For each $\ell \in [m]$ we have obtained an approval scoring function f_ℓ which defines \mathcal{F} on ℓ -regular profiles. Our goal is to show that there exists a linear combination of these approvals scoring functions $f = \sum_{\ell \in [m]} \gamma_\ell f_\ell$ which defines \mathcal{F} on arbitrary profiles. We define the corresponding coefficients $\gamma_1, \dots, \gamma_m$ inductively. We first construct two specific committees W_1^* and W_2^* , which we use to scale the coefficients, and additionally, in order to define coefficient $\gamma_{\ell+1}$ we construct two specific votes, $a_{\ell+1}^*$ and $b_{\ell+1}^*$, with exactly $\ell + 1$, and at most ℓ approved candidates, respectively. We define coefficient $\gamma_{\ell+1}$ using the definition of f for ℓ -bounded profiles and by exploring how \mathcal{F} compares committees W_1^* and W_2^* for very specific profiles which are built from certain numbers of votes $a_{\ell+1}^*$ and $b_{\ell+1}^*$. This concludes the construction of f .

Showing that $f = \sum_{\ell \in [m]} \gamma_\ell f_\ell$ implements \mathcal{F} requires a rather involved analysis, which is divided into several lemmas. In Lemma 7 we show that f implements \mathcal{F} , but only for the case when \mathcal{F} is used to compare W_1^* and W_2^* , and only for very specific profiles. In Lemma 8 we still assume that \mathcal{F} is used to compare only W_1^* and W_2^* , but this time we extend the statement to arbitrary profiles. In Lemma 10 we show the case when \mathcal{F} is used to compare W_1^* with any other committee. We complete this reasoning with a short discussion explaining the validity of our statement in its full generality. Each of the aforementioned lemmas is based on a different idea and they build upon each other. The main proof technique is to transform simple approval profiles to more complex ones and argue that certain properties are preserved due to the required axioms. An overview of the proof structure is displayed in Fig. A.2.

A.1. A stronger efficiency axiom

In the subsequent proofs we will use the following *strong efficiency* axiom:

Strong Efficiency. An ABC ranking rule \mathcal{F} satisfies *strong efficiency* if for every voter set $V \in \mathcal{V}$, committees $W_1, W_2 \in \mathcal{P}_k(C)$ and $A \in \mathcal{A}(V)$ where for every voter $v \in V$ we have $|A(v) \cap W_1| \geq |A(v) \cap W_2|$, it holds that $W_1 \succeq_{\mathcal{F}(A)} W_2$.

For $k = 1$, i.e., in the single-winner setting, strong efficiency is the well-known Pareto efficiency axiom, which requires that if a candidate c is unanimously preferred to candidate d , then $d \succeq c$ in the collective ranking (Moulin, 1988).

The following lemma shows that strong efficiency in the context of neutral and consistent rules is implied by its weaker counterpart.

Lemma 2. *An ABC ranking rule that satisfies neutrality, consistency and weak efficiency also satisfies strong efficiency.*

Proof. Let \mathcal{F} be an ABC ranking rule that satisfies neutrality, consistency and weak efficiency. Further, let $W_1, W_2 \in \mathcal{P}_k(C)$ and $A \in \mathcal{A}(V)$ such that for every vote $v \in V$ we have $|A(v) \cap W_1| \geq |A(v) \cap W_2|$. We have to show that $W_1 \succeq_{\mathcal{F}(A)} W_2$. Fix $v \in V$ and let $A_v \in \mathcal{A}(\{1\})$ be the profile containing the single vote $A(v)$. Now, let us consider a committee W'_2 constructed from W_2 in the following way. We obtain W'_2 from W_2 by replacing candidates in $W_2 \setminus A(v)$ with candidates from $A(v)$ so that $|A(v) \cap W'_2| = |A(v) \cap W_1|$. Note that $A(v) \cap W_2 \subseteq A(v) \cap W'_2$ and hence candidates in $A(v) \cap (W_2 \setminus W'_2) = \emptyset$. Hence by weak efficiency we get that $W'_2 \succeq_{\mathcal{F}(A_v)} W_2$. Furthermore, neutrality implies that $W'_2 \sim_{\mathcal{F}(A_v)} W_1$ and by transitivity we infer that $W_1 \succeq_{\mathcal{F}(A_v)} W_2$. The final step is to apply consistency. For every $v \in V$, $W_1 \succeq_{\mathcal{F}(A_v)} W_2$. Hence also for their combination $\sum_{v \in V} A_v = A$ we have $W_1 \succeq_{\mathcal{F}(A)} W_2$. \square

A.2. \mathcal{F} is an ABC scoring rule on ℓ -regular approval profiles

Recall that we assume that \mathcal{F} is an ABC ranking rule satisfying symmetry, consistency, weak efficiency, and continuity. If \mathcal{F} is trivial, i.e., if \mathcal{F} always maps to the trivial relation, then \mathcal{F} is the ABC scoring rule implemented by $f(x, y) = 0$. Thus, hereinafter we assume that \mathcal{F} is a fixed, non-trivial ABC ranking rule satisfying anonymity, neutrality, weak efficiency, and continuity. By Lemma 2 we can also assume that \mathcal{F} satisfies strong efficiency.

As a first step, we will prove in this section that \mathcal{F} restricted to ℓ -regular approval profiles is an ABC scoring rule, i.e., that there exists an approval scoring function that implements \mathcal{F} on ℓ -regular approval profiles.

For each $\ell \in [m]$, from \mathcal{F} we construct an LOC ranking rule, \mathcal{G}_ℓ , as follows. For a profile of linear orders P , by $Appr(P, \ell)$ we denote the approval preference profile where voters approve of their top ℓ candidates. We define for every $\ell \in [m]$ an LOC ranking rule \mathcal{G}_ℓ , as:

$$\mathcal{G}_\ell(P) = \mathcal{F}(Appr(P, \ell)). \tag{A.1}$$

Lemma 3, below, shows that our construction preserves the axioms under consideration and consequently that \mathcal{G}_ℓ is an LOC scoring rule. As mentioned before, this lemma heavily builds upon the characterization result of Skowron et al. (2019).

Lemma 3. *Let \mathcal{F} be an ABC ranking rule satisfying symmetry, consistency, strong efficiency and continuity. Then for each $\ell \in [m]$, the LOC ranking rule \mathcal{G}_ℓ defined by Equation (A.1) is an LOC scoring rule.*

Proof. The proof of this lemma relies on the main theorem of Skowron et al. (2019): an LOC ranking rule is a LOC scoring rule if and only if it satisfies anonymity, neutrality, consistency, committee dominance, and continuity. We thus have to verify that \mathcal{G}_ℓ satisfies these axioms. Note that since \mathcal{G}_ℓ is an LOC ranking rule, the corresponding axioms differ slightly from the ones introduced in Section 3.2. Thus, in the following we introduce each of these axioms for LOC ranking rules and prove that it is satisfied by \mathcal{G}_ℓ for arbitrary ℓ .

(Anonymity) An LOC ranking rule \mathcal{G} satisfies anonymity if for every two sets of voters $V, V' \in \mathcal{V}$ such that $|V| = |V'|$, for each bijection $\rho : V \rightarrow V'$ and for every two profiles of linear orders P_1 and P_2 over V and V' , respectively, such that $P_1(v) = P_2(\rho(v))$ for each $v \in V$, it holds that $\mathcal{G}(P_1) = \mathcal{G}(P_2)$. Let V, V', ρ, P_1, P_2 be defined accordingly. Note that $P_1(v) = P_2(\rho(v))$ implies that $Appr(P_1, \ell)(v) = Appr(P_2, \ell)(\rho(v))$. Hence, by anonymity of \mathcal{F} ,

$$\mathcal{G}_\ell(P_1) = \mathcal{F}(Appr(P_1, \ell)) = \mathcal{F}(Appr(P_2, \ell)) = \mathcal{G}_\ell(P_2).$$

(Neutrality) An LOC ranking rule \mathcal{G} satisfies neutrality if for each permutation σ of C and every two preference profiles P_1, P_2 over the same voter set V with $P_1 = \sigma(P_2)$, it holds that $\mathcal{G}(P_1) = \sigma(\mathcal{G}(P_2))$. Let P_1, P_2, V , and σ be defined accordingly. Note that $Appr(P_1, \ell) = \sigma(Appr(P_2, \ell))$. Then, by neutrality of \mathcal{F} ,

$$\mathcal{G}_\ell(P_1) = \mathcal{F}(Appr(P_1, \ell)) = \mathcal{F}(\sigma(Appr(P_2, \ell))) = \sigma(\mathcal{F}(Appr(P_2, \ell))) = \sigma(\mathcal{G}_\ell(P_2)).$$

(Consistency) An LOC ranking rule \mathcal{G} satisfies consistency if for every two profiles P_1 and P_2 over disjoint sets of voters, $V \in \mathcal{V}$ and $V' \in \mathcal{V}$, $V \cap V' = \emptyset$, and every two committees $W_1, W_2 \in \mathcal{P}_k(C)$, (i) if $W_1 \succ_{\mathcal{G}(P_1)} W_2$ and $W_1 \succeq_{\mathcal{G}(P_2)} W_2$, then it holds that $W_1 \succ_{\mathcal{G}(P_1+P_2)} W_2$ and (ii) if $W_1 \succeq_{\mathcal{G}(P_1)} W_2$ and $W_1 \succeq_{\mathcal{G}(P_2)} W_2$, then it holds that $W_1 \succeq_{\mathcal{G}(P_1+P_2)} W_2$. Let P_1, P_2, V, V', W_1 , and W_2 be defined accordingly. Let us prove (i). If $W_1 \succ_{\mathcal{G}_\ell(P_1)} W_2$, then $W_1 \succ_{\mathcal{F}(Appr(P_1, \ell))} W_2$. Analogously, if $W_1 \succeq_{\mathcal{G}_\ell(P_2)} W_2$, then $W_1 \succeq_{\mathcal{F}(Appr(P_2, \ell))} W_2$. By consistency of \mathcal{F} , we know that $W_1 \succ_{\mathcal{F}(Appr(P_1, \ell)+Appr(P_2, \ell))} W_2$. Clearly, $Appr(P_1, \ell) + Appr(P_2, \ell) = Appr(P_1 + P_2, \ell)$. We can conclude that $W_1 \succ_{\mathcal{F}(Appr(P_1+P_2, \ell))} W_2$ and hence $W_1 \succ_{\mathcal{G}_\ell(P_1+P_2)} W_2$. The proof of (ii) is analogous.

(Committee dominance) An LOC ranking rule \mathcal{G} satisfies committee dominance if for every two committees $W_1, W_2 \in \mathcal{P}_k(C)$ and each profile of linear orders P where for every vote $v \in V$, $\text{pos}_v(W_1, P)$ dominates $\text{pos}_v(W_2, P)$, it holds that $W_1 \succeq_{\mathcal{G}(P)} W_2$. Let W_1, W_2 , and P be defined accordingly. If $\text{pos}_v(W_1, P)$ dominates $\text{pos}_v(W_2, P)$, then clearly for each $v \in V$, $|Appr(P, \ell)(v) \cap W_1| \geq |Appr(P, \ell)(v) \cap W_2|$. By strong efficiency of \mathcal{F} , $W_1 \succeq_{\mathcal{G}_\ell(P)} W_2$.

(Continuity) An LOC ranking rule \mathcal{G} satisfies continuity if for every two committees $W_1, W_2 \in \mathcal{P}_k(C)$ and every two profiles P_1 and P_2 where $W_1 \succ_{\mathcal{G}(P_2)} W_2$, there exists a number $n \in \mathbb{N}$ such that $W_1 \succ_{\mathcal{G}(P_1+nP_2)} W_2$. This is an immediate consequence of the fact that \mathcal{F} satisfies continuity. \square

Lemma 3 shows that there exists a committee scoring function implementing rule \mathcal{G}_ℓ . The following lemma shows that this committee scoring function has a special form that allows it to be represented by an approval scoring function.

Lemma 4. For $\ell \in [m]$, let $g_\ell : \mathcal{P}_k([m]) \rightarrow \mathbb{R}$ be a committee scoring function that implements \mathcal{G}_ℓ . There exists an approval scoring function f_ℓ such that:

$$g_\ell(I) = f_\ell(|\{i \in I : i \leq \ell\}|, \ell) \quad \text{for each } I \in [m]_k \text{ and } \ell \in [m].$$

Proof. We have to show that for an arbitrary profile of linear orders P over V and some $v \in V$, two committees W_1 and W_2 have the same score $g_\ell(\text{pos}_v(W_1)) = g_\ell(\text{pos}_v(W_2))$ given that

$$|\{i \in \text{pos}_v(W_1) : i \leq \ell\}| = |\{i \in \text{pos}_v(W_2) : i \leq \ell\}|.$$

From the neutrality of \mathcal{F} , we see that if v has the same number of approved members in W_1 as in W_2 , W_1 and W_2 are equally good with respect to \mathcal{F} . Thus if W_1 and W_2 have the same number of members in the top ℓ positions in v , then W_1 and W_2 are also equally good with respect to \mathcal{G}_ℓ . Hence the scores assigned by g_ℓ to the positions occupied by W_1 and W_2 are the same. \square

We are now ready to prove Lemma 5, which provides the main technical conclusion of this section.

Lemma 5. For each $\ell \in [m]$, the approval scoring function $f_\ell(a, \ell)$, as defined in the statement of Lemma 4, implements \mathcal{F} on ℓ -regular approval profiles.

Proof. For each ℓ -regular approval profile A we can create an ordinal profile $\text{Rank}(A, \ell)$ where voters put all approved candidates in their top ℓ positions (in some fixed arbitrary order) and in the next $(m - \ell)$ positions the candidates that they disapprove of (also in some fixed arbitrary order). Naturally, $\text{Appr}(\text{Rank}(A, \ell), \ell) = A$. Thus, a committee W_1 is preferred over W_2 in A according to \mathcal{F} if and only if W_1 is preferred over W_2 in $\text{Rank}(A, \ell)$ according to \mathcal{G}_ℓ . Since \mathcal{G}_ℓ is an LOC scoring rule, the previous statement holds if and only if W_1 has a higher score than W_2 according to the committee scoring function g_ℓ . This is equivalent to W_1 having a higher score according to f_ℓ (Lemma 4). We conclude that W_1 is preferred over W_2 in A according to \mathcal{F} if and only if W_1 has a higher score according to f_ℓ . Consequently, we have shown that \mathcal{F} is an ABC scoring rule for ℓ -regular approval profiles. \square

As the construction in the proof of Lemma 5 relies on $\text{Rank}(A, \ell)$ and so it applies only to profiles where each voter approves the same number of candidates, we need new ideas to prove that \mathcal{F} is an ABC scoring rule on arbitrary profiles. We explain these ideas in the following section.

A.3. \mathcal{F} is an ABC scoring rule on arbitrary profiles

We now generalize the result of Lemma 5 for ℓ -regular profiles to arbitrary approval profiles. We will use here the following notation.

Definition 4. For an approval profile $A \in \mathcal{A}(V)$ and $\ell \in [0, m]$ we write $\text{Bnd}(A, \ell)$ to denote the profile consisting of all votes $v \in V$ with $A(v) \leq \ell$, i.e., $\text{Bnd}(A, \ell) \in \mathcal{A}(V')$ with $V' = \{v \in V : A(v) \leq \ell\}$ and $\text{Bnd}(A, \ell)(v) = A(v)$ for all $v \in V'$. Analogously, we write $\text{Reg}(A, \ell)$ to denote the profile consisting of all votes $A(v)$, for $v \in V$ with $A(v) = \ell$.

Clearly, $\text{Bnd}(A, \ell)$ is ℓ -bounded and $\text{Reg}(A, \ell)$ is ℓ -regular.

Now, let $\{f_\ell\}_{\ell \leq m}$ be the family of approval scoring functions witnessing that \mathcal{F} , when applied to ℓ -regular profiles, is an ABC scoring rule (cf. Lemma 5). From $\{f_\ell\}_{\ell \leq m}$ we will now construct a single approval scoring function f that witnesses that \mathcal{F} is an ABC scoring rule. Since f and f_ℓ have to produce the same output on ℓ -regular profiles, it would be tempting to define $f(x, \ell) = f_\ell(x, \ell)$. However, this simple construction does not work. Instead, we will find constants $\gamma_1, \dots, \gamma_m$ such that $f(x, \ell) = \gamma_\ell \cdot f_\ell(x, \ell)$ and show that with this construction we indeed obtain an approval scoring function implementing \mathcal{F} .

For this construction, let us fix two arbitrary committees W_1^*, W_2^* with the smallest possible size of the intersection. In particular, $W_1^* \cap W_2^* = \emptyset$ for $m \geq 2k$. Let $W_1^* \setminus W_2^* = \{a_1, \dots, a_t\}$, and let $W_2^* \setminus W_1^* = \{b_1, \dots, b_t\}$. By σ^* we denote the permutation that swaps a_1 with b_1 , a_2 with b_2 , etc., and that is the identity elsewhere.

We will define $\gamma_1, \dots, \gamma_m$ inductively. For the base case we set $f(0, 0) = 0$. Now, let us assume that f is defined on $[0, k] \times [0, \ell]$ and that f implements \mathcal{F} on ℓ -bounded profiles. To choose $\gamma_{\ell+1}$, we distinguish the following three cases:

- Case (A).** If in all $(\ell + 1)$ -regular profiles A it holds that $W_1^* \sim_{\mathcal{F}(A)} W_2^*$, then we set $\gamma_{\ell+1} = 0$.
- Case (B).** If we are not in Case (A) and in all ℓ -bounded profiles A it holds that $W_1^* \sim_{\mathcal{F}(A)} W_2^*$, then we set $\gamma_{\ell+1} = 1$.
- Case (C).** Otherwise, there exist a single-vote $(\ell + 1)$ -regular profile A such that $W_1^* \succ_{\mathcal{F}(A)} W_2^*$ and a single-vote ℓ -bounded profile A' such that $W_1^* \sim_{\mathcal{F}(A')} W_2^*$. Indeed, if for all $(\ell + 1)$ -regular single-vote profiles $A \in \mathcal{A}(\{1\})$ it holds that $W_1^* \sim_{\mathcal{F}(A)} W_2^*$, then by consistency this holds for all $(\ell + 1)$ -regular profiles, which is a precondition of Case (A). Similarly, if for all ℓ -bounded single-vote profiles $A \in \mathcal{A}(\{1\})$ it holds that $W_1^* \sim_{\mathcal{F}(A)} W_2^*$, then by consistency this holds for all ℓ -bounded profiles (precondition of Case (B)). Consequently, the profiles A and A' can be chosen to consist of single votes.

In the following, by slight abuse of notation, we identify a set of approved candidates with its corresponding single-vote profile. Let $a_{\ell+1}^* \subseteq C$ be a vote such that (i) $|a_{\ell+1}^*| = \ell + 1$, (ii) $W_1^* \succ_{\mathcal{F}(a_{\ell+1}^*)} W_2^*$, and (iii) such that the difference between the scores of W_1^* and W_2^* is maximized. Furthermore, let $b_{\ell+1}^* \subseteq C$ be a vote such that (i) $|b_{\ell+1}^*| \leq \ell$, (ii) $W_1^* \succ_{\mathcal{F}(b_{\ell+1}^*)} W_2^*$, and (iii) such that the difference between the scores of W_1^* and W_2^* is maximized. For each $x, y \in \mathbb{N}$ we define the profile $S(x, y)$ as:

$$S(x, y) = x \cdot \sigma^*(a_{\ell+1}^*) + y \cdot b_{\ell+1}^*.$$

Let us define $t_{\ell+1}^*$ as:

$$t_{\ell+1}^* = \sup \left\{ \frac{x}{y} : W_1^* \succ_{\mathcal{F}(S(x,y))} W_2^* \right\}, \tag{A.2}$$

which is a well-defined positive real number as we show in Lemma 6. We define:

$$\gamma_{\ell+1} = \frac{\text{sc}_f(W_1^*, b_{\ell+1}^*) - \text{sc}_f(W_2^*, b_{\ell+1}^*)}{t_{\ell+1}^* \cdot \left(\text{sc}_{f_{\ell+1}}(W_1^*, a_{\ell+1}^*) - \text{sc}_{f_{\ell+1}}(W_2^*, a_{\ell+1}^*) \right)}.$$

This concludes the construction of f . Let us now show that $t_{\ell+1}^*$ is a positive real number and that it defines a threshold:

Lemma 6. *The supremum $t_{\ell+1}^*$, as defined by Equation (A.2), is a positive real number. Furthermore, if $x/y < t_{\ell+1}^*$, then $W_1^* \succ_{\mathcal{F}(S(x,y))} W_2^*$. If $x/y > t_{\ell+1}^*$, then $W_2^* \succ_{\mathcal{F}(S(x,y))} W_1^*$.*

Proof. Let us argue that $t_{\ell+1}^*$ is well defined. By continuity there exists y such that $W_1^* \succ_{\mathcal{F}(S(1,y))} W_2^*$. Consequently, the set in (A.2) is nonempty. Also by continuity, there exists x such that $W_2^* \succ_{\mathcal{F}(S(x,1))} W_1^*$. Further, we observe that for each x', y' with $x'/y' > x$ it also holds that $W_2^* \succ_{\mathcal{F}(S(x',y'))} W_1^*$. Indeed, since $S(x', y') = S(xy', y) + S(x' - xy', 0)$, we infer that in such case $S(x', y')$ can be split into y' copies of $S(x, 1)$ and $x' - xy'$ copies of $\sigma^*(a_{\ell+1}^*)$. By consistency we get $W_2^* \succ_{\mathcal{F}(S(x',y'))} W_1^*$. Thus, the set in (A.2) is bounded, and so $t_{\ell+1}^*$ is a positive real number.

To show the second statement, let us assume that $x/y < t_{\ell+1}^*$. From the definition of $t_{\ell+1}^*$ we infer that there exist $x', y' \in \mathbb{N}$, such that $x/y < x'/y'$ and such that $W_1^* \succ_{\mathcal{F}(S(x',y'))} W_2^*$. By consistency, it also holds that $W_1^* \succ_{\mathcal{F}(S(xx',xy'))} W_2^*$. Since $W_1^* \succ_{\mathcal{F}(S(0,1))} W_2^*$ and $x'y - xy' > 0$ and we get that $W_1^* \succ_{\mathcal{F}(S(0,x'y-xy'))} W_2^*$. Now, observe that

$$S(xx', x'y) = S(xx', xy') + S(0, x'y - xy').$$

Thus, from consistency infer that $W_1^* \succ_{\mathcal{F}(S(xx',x'y))} W_2^*$. Again, by consistency we get that $W_1^* \succ_{\mathcal{F}(S(x,y))} W_2^*$.

Next, let us assume that $x/y > t_{\ell+1}^*$. Then, there exist $x', y' \in \mathbb{N}$, such that $x/y > x'/y'$ and such that $W_2^* \succ_{\mathcal{F}(S(x',y'))} W_1^*$. Similarly as before, we get that $W_2^* \succ_{\mathcal{F}(S(x'y,yy'))} W_1^*$ and since $xy' - x'y > 0$ we get that $W_2^* \succ_{\mathcal{F}(S(xy'-x'y,0))} W_1^*$. Since $S(xy', yy') = S(x'y, yy') + S(xy' - x'y, 0)$, consistency implies that $W_2^* \succ_{\mathcal{F}(S(xy',yy'))} W_1^*$. Finally, we get that $W_2^* \succ_{\mathcal{F}(S(x,y))} W_1^*$, which completes the proof. \square

In the remainder of this section, we prove that f is indeed an approval scoring function that implements \mathcal{F} and thus \mathcal{F} is an ABC scoring rule. We prove this for increasingly general profiles, starting with very simple ones, and at first we prove a slightly weaker relation between f and \mathcal{F} .

Lemma 7. *Let us fix $\ell \in [m - 1]$. Let $A \in \mathcal{A}(V)$ be an approval profile with $A(v) \in \{a_{\ell+1}^*, b_{\ell+1}^*, \sigma^*(a_{\ell+1}^*), \sigma^*(b_{\ell+1}^*)\}$ for all $v \in V$. Then:*

$$sc_f(W_1^*, A) > sc_f(W_2^*, A) \implies W_1^* \succ_{\mathcal{F}(A)} W_2^*.$$

Proof. We start by noting that if $b_{\ell+1}^*$ and $a_{\ell+1}^*$ are defined, then Case (C) occurred when defining $\gamma_{\ell+1}$. In particular, $t_{\ell+1}^*$ has been defined and Lemma 6 is applicable.

First we show that if A contains both $a_{\ell+1}^*$ and $\sigma^*(a_{\ell+1}^*)$, then after removing both from A the relative order of W_1^* and W_2^* does not change. Without loss of generality, let us assume that $W_1^* \succ_{\mathcal{F}(A)} W_2^*$ and consider the profile Q that consist of two votes, $a_{\ell+1}^*$ and $\sigma^*(a_{\ell+1}^*)$. By neutrality, W_1^* and W_2^* are equally good with respect to Q . If $W_2^* \succeq_{\mathcal{F}(A-Q)} W_1^*$, then by consistency we would get that $W_2^* \succeq_{\mathcal{F}(A)} W_1^*$, a contradiction. By the same argument we observe that if A contains $b_{\ell+1}^*$ and $\sigma^*(b_{\ell+1}^*)$, then after removing them from A the relative order of W_1^* and W_2^* does not change. Further if A contains only votes $b_{\ell+1}^*$ and $a_{\ell+1}^*$, then by consistency we can infer that W_1^* is preferred over W_2^* in A . Also, A cannot contain only votes $\sigma^*(b_{\ell+1}^*)$ and $\sigma^*(a_{\ell+1}^*)$, since in both these single-vote profiles the score of W_2^* is greater than the score of W_1^* (this follows from Lemma 5 and from the fact that f for ℓ -regular profiles is a linear transformation of an appropriate approval scoring function f_ℓ).

The above reasoning shows that without loss of generality we can assume that in A there are either only the votes of types $b_{\ell+1}^*$ and $\sigma^*(a_{\ell+1}^*)$ or only the votes of types $a_{\ell+1}^*$ and $\sigma^*(b_{\ell+1}^*)$. Let us consider the first case, and let us assume that in A there are y_A votes of type $b_{\ell+1}^*$ and x_A votes of type $\sigma^*(a_{\ell+1}^*)$. Since $sc_f(W_1^*, A) > sc_f(W_2^*, A)$, we get that:

$$\begin{aligned}
 & y_A \cdot \text{sc}_f(W_1^*, b_{\ell+1}^*) + x_A \cdot \text{sc}_f(W_1^*, \sigma^*(a_{\ell+1}^*)) \\
 & > y_A \cdot \text{sc}_f(W_2^*, b_{\ell+1}^*) + x_A \cdot \text{sc}_f(W_2^*, \sigma^*(a_{\ell+1}^*)).
 \end{aligned}$$

Thus, from the definition of σ^* we get that:

$$y_A \cdot \text{sc}_f(W_1^*, b_{\ell+1}^*) + x_A \cdot \text{sc}_f(W_2^*, a_{\ell+1}^*) > y_A \cdot \text{sc}_f(W_2^*, b_{\ell+1}^*) + x_A \cdot \text{sc}_f(W_1^*, a_{\ell+1}^*).$$

Which is equivalent to:

$$x_A \cdot (\text{sc}_f(W_1^*, a_{\ell+1}^*) - \text{sc}_f(W_2^*, a_{\ell+1}^*)) < y_A \cdot (\text{sc}_f(W_1^*, b_{\ell+1}^*) - \text{sc}_f(W_2^*, b_{\ell+1}^*)).$$

From the above inequality we get that:

$$\begin{aligned}
 \frac{x_A}{y_A} & < \frac{\text{sc}_f(W_1^*, b_{\ell+1}^*) - \text{sc}_f(W_2^*, b_{\ell+1}^*)}{\text{sc}_f(W_1^*, a_{\ell+1}^*) - \text{sc}_f(W_2^*, a_{\ell+1}^*)} = \frac{\text{sc}_f(W_1^*, b_{\ell+1}^*) - \text{sc}_f(W_2^*, b_{\ell+1}^*)}{\gamma_{\ell+1}(\text{sc}_{f_{\ell+1}}(W_1^*, a_{\ell+1}^*) - \text{sc}_{f_{\ell+1}}(W_2^*, a_{\ell+1}^*))} \\
 & = t_{\ell+1}^*.
 \end{aligned}$$

Observe that $A = S(x_A, y_A)$, so since $x_A/y_A < t_{\ell+1}^*$, from Lemma 6 we infer that $W_1^* \succ_{\mathcal{F}(A)} W_2^*$.

Now, let us assume that A consists only of the votes of types $a_{\ell+1}^*$ and $\sigma^*(b_{\ell+1}^*)$. In such case the profile $\sigma^*(A)$ consists only of votes of types $b_{\ell+1}^*$ and $\sigma^*(a_{\ell+1}^*)$. Further, $\text{sc}_f(W_2^*, \sigma^*(A)) > \text{sc}_f(W_1^*, \sigma^*(A))$. Similarly as before, let us assume that in $\sigma^*(A)$ there are y_A votes of type $b_{\ell+1}^*$ and x_A votes of type $\sigma^*(a_{\ell+1}^*)$. By similar reasoning as before we infer that $x_A/y_A > t_{\ell+1}^*$, and by Lemma 6 that $W_2^* \succ_{\mathcal{F}(\sigma^*(A))} W_1^*$. From this, by neutrality, it follows that $W_1^* \succ_{\mathcal{F}(A)} W_2^*$, which completes the proof. \square

Next, we generalize Lemma 7 to arbitrary profiles, yet we still focus on comparing the two distinguished profiles W_1^* and W_2^* .

Lemma 8. For all $A \in \mathcal{A}(V)$ it holds that

$$\text{sc}_f(W_1^*, A) > \text{sc}_f(W_2^*, A) \implies W_1^* \succ_{\mathcal{F}(A)} W_2^*.$$

Proof. We prove this statement by induction on ℓ -bounded profiles. For 0-bounded profiles A this is trivial since $\text{sc}_f(W_1^*, A) > \text{sc}_f(W_2^*, A)$ cannot hold.

Assume that the statement holds for ℓ -bounded profiles and assume that $\text{sc}_f(W_1^*, A) > \text{sc}_f(W_2^*, A)$. If Case (A) was applicable when defining $\gamma_{\ell+1}$, i.e., if $\gamma_{\ell+1} = 0$, then $\text{sc}_f(W_1^*, A) > \text{sc}_f(W_2^*, A)$ implies $\text{sc}_f(W_1^*, \text{Bnd}(A, \ell)) > \text{sc}_f(W_2^*, \text{Bnd}(A, \ell))$ since the score of $(\ell + 1)$ -regular profiles is 0. This implies by the induction hypothesis that $W_1^* \succ_{\mathcal{F}(\text{Bnd}(A, \ell))} W_2^*$. Furthermore, since Case (A) was applicable, $W_1^* \sim_{\mathcal{F}(\text{Reg}(A, \ell+1))} W_2^*$. Since $A = \text{Bnd}(A, \ell) + \text{Reg}(A, \ell + 1)$, consistency yields that $W_1^* \succ_{\mathcal{F}(A)} W_2^*$.

In Case (B), we know that $W_1^* \sim_{\mathcal{F}(A)} W_2^*$ for all ℓ -bounded profiles. Hence $W_1^* \sim_{\mathcal{F}(\text{Bnd}(A, \ell))} W_2^*$. By our induction hypothesis, this implies that $\text{sc}_f(W_1^*, \text{Bnd}(A, \ell)) = \text{sc}_f(W_2^*, \text{Bnd}(A, \ell))$. Hence $\text{sc}_f(W_1^*, \text{Reg}(A, \ell + 1)) > \text{sc}_f(W_2^*, \text{Reg}(A, \ell + 1))$. Recall that Lemma 5 states that $f_{\ell+1}$ implements \mathcal{F} on $(\ell + 1)$ -regular profiles. Since $\text{Reg}(A, \ell + 1)$ is an $(\ell + 1)$ -regular profile and $f(x, \ell + 1) = f_{\ell+1}(x, \ell + 1)$, in particular $\text{sc}_f(W_1^*, \text{Reg}(A, \ell + 1)) > \text{sc}_f(W_2^*, \text{Reg}(A, \ell + 1))$ implies $W_1^* \succ_{\mathcal{F}(\text{Reg}(A, \ell+1))} W_2^*$. Furthermore, by consistency, W_1^* has the same relative position as W_2^* in $\mathcal{F}(\text{Reg}(A, \ell + 1))$ and $\mathcal{F}(A)$, which in turn implies $W_1^* \succ_{\mathcal{F}(A)} W_2^*$.

In Case (C), for the sake of contradiction let us assume that $W_2^* \succeq_{\mathcal{F}(A)} W_1^*$. Let us take an arbitrary vote $v \in V$ with $A(v) \notin \{b_{\ell+1}^*, a_{\ell+1}^*, \sigma^*(b_{\ell+1}^*), \sigma^*(a_{\ell+1}^*)\}$. We will show in the following that there exists a profile A' with $\text{set}(A') = \text{set}(A) \setminus \{A(v)\}$, $\text{sc}_f(W_1^*, A') >$

$sc_f(W_2^*, A')$, and $W_2^* \succeq_{\mathcal{F}(A')} W_1^*$. We then repeat this step until we obtain a profile A'' with $set(A'') = \{b_{\ell+1}^*, a_{\ell+1}^*, \sigma^*(b_{\ell+1}^*), \sigma^*(a_{\ell+1}^*)\}$. Still, it holds that $sc_f(W_1^*, A'') > sc_f(W_2^*, A'')$ and $W_2^* \succeq_{\mathcal{F}(A'')} W_1^*$, but that contradicts Lemma 7. Consequently, $W_1^* \succ_{\mathcal{F}(A)} W_2^*$ has to hold.

Let us now show that there exists a profile A' with $set(A') = set(A) \setminus \{A(v)\}$, $sc_f(W_1^*, A') > sc_f(W_2^*, A')$, and $W_2^* \succeq_{\mathcal{F}(A')} W_1^*$. If $W_1^* \sim_{\mathcal{F}(A(v))} W_2^*$, then by consistency the relative order of W_1^* and W_2^* in $\mathcal{F}(A')$ is the same as in $\mathcal{F}(A)$. Also, since the scores of committees W_1^* and W_2^* are the same in v (cf. Lemma 5), we get that $sc_f(W_1^*, A') > sc_f(W_2^*, A')$.

Let us now consider the case that $W_1^* \succ_{\mathcal{F}(A(v))} W_2^*$. Let $n_v = |\{v' \in V : A(v') = A(v)\}|$. We set

$$\epsilon = sc_f(W_1^*, A) - sc_f(W_2^*, A) > 0. \tag{A.3}$$

We distinguish two cases: $|A(v)| \leq \ell$ and $|A(v)| = \ell + 1$. Let us consider $|A(v)| \leq \ell$ first. We observe that there exist values $x, y \in \mathbb{N}$ such that:

$$0 < \frac{x}{y}(sc_f(W_1^*, \sigma^*(b_{\ell+1}^*)) - sc_f(W_2^*, \sigma^*(b_{\ell+1}^*))) + n_v(sc_f(W_1^*, v) - sc_f(W_2^*, v)) < \frac{\epsilon}{2}. \tag{A.4}$$

Now, consider a profile $B = y \cdot A + x \cdot \sigma^*(b_{\ell+1}^*) + x \cdot b_{\ell+1}^*$. By consistency, $W_2^* \succeq_{\mathcal{F}(B)} W_1^*$. Next, let us consider a profile $Q = x \cdot \sigma^*(b_{\ell+1}^*) + y \cdot n_v \cdot A(v)$. From Equality (A.4) we see that W_1^* has a higher score in Q than W_2^* . Since Q is ℓ -bounded, by our inductive assumption we get that $W_1^* \succ_{\mathcal{F}(Q)} W_2^*$. Consequently, by consistency we get that $W_2^* \succ_{\mathcal{F}(B-Q)} W_1^*$ since otherwise $W_1^* \succ_{\mathcal{F}(B)} W_2^*$, a contradiction. Further, from Equalities (A.3) and (A.4) we get that in $B - Q$ the score of W_1^* is greater than the score of W_2^* , which can be seen as follows:

$$\begin{aligned} &sc_f(W_1^*, B - Q) - sc_f(W_2^*, B - Q) \\ &= sc_f(W_1^*, B) - sc_f(W_2^*, B) - (sc_f(W_1^*, Q) - sc_f(W_2^*, Q)) \\ &= y\epsilon - (sc_f(W_1^*, Q) - sc_f(W_2^*, Q)) > \frac{y\epsilon}{2}. \end{aligned}$$

We obtained the profile $B - Q = y \cdot A + x(\sigma^*(b_{\ell+1}^*) + b_{\ell+1}^*) - x \cdot \sigma^*(b_{\ell+1}^*) - y \cdot n_v \cdot A(v) = y \cdot (A - n_v \cdot A(v)) + x \cdot b_{\ell+1}^*$, for which $set(B - Q) = set(A) \setminus \{A(v)\}$. Furthermore, the relative order of W_1^* and W_2^* in $\mathcal{F}(B - Q)$ is the same as in $\mathcal{F}(A)$, and $sc_f(W_1^*, B - Q) > sc_f(W_2^*, B - Q)$.

Let us now turn to the case that $|A(v)| = \ell + 1$. Similar to before, we choose $x, y \in \mathbb{N}$ such that:

$$0 < \frac{x}{y}(sc_f(W_1^*, \sigma^*(a_{\ell+1}^*)) - sc_f(W_2^*, \sigma^*(a_{\ell+1}^*))) + n_v(sc_f(W_1^*, v) - sc_f(W_2^*, v)) < \frac{\epsilon}{2}. \tag{A.5}$$

Now, consider a profile $B = y \cdot A + x \cdot \sigma^*(a_{\ell+1}^*) + x \cdot a_{\ell+1}^*$ for which, by consistency, $W_2^* \succeq_{\mathcal{F}(B)} W_1^*$ holds. Let $Q = x \cdot \sigma^*(a_{\ell+1}^*) + y \cdot n_v \cdot A(v)$. From Equality (A.5) we see that W_1^* has a higher score in Q than W_2^* . Since Q is $(\ell + 1)$ -regular, Lemma 5 gives us that $W_1^* \succ_{\mathcal{F}(Q)} W_2^*$. As before, by consistency we get that $W_2^* \succ_{\mathcal{F}(B-Q)} W_1^*$, and from Equalities (A.3) and (A.5) we get that $sc_f(W_1^*, B - Q) > sc_f(W_2^*, B - Q)$. Hence, also in this case, we have obtained the profile $B - Q$, for which $set(B - Q) = set(A) \setminus \{A(v)\}$, the relative order of W_1^* and W_2^* in $\mathcal{F}(B - Q)$ is the same as in $\mathcal{F}(A)$, and $sc_f(W_1^*, B - Q) > sc_f(W_2^*, B - Q)$.

Finally, if $W_2^* \succ_{\mathcal{F}(A(v))} W_1^*$ in v , we can repeat the above reasoning, but applying σ^* to all occurrences of $b_{\ell+1}^*, a_{\ell+1}^*, \sigma^*(b_{\ell+1}^*)$, and $\sigma^*(a_{\ell+1}^*)$. \square

Before we proceed further, we establish the existence of two particular profiles A_ℓ^* and B_ℓ^* , that we will need for proving the most general variant of our statement.

Lemma 9. *Let $W_1, W_2, W_3 \in \mathcal{P}_k(C)$ such that $|W_1 \cap W_3| > |W_1 \cap W_2|$. For each $\ell, 1 \leq \ell \leq m$, if \mathcal{F} is non-trivial for ℓ -regular profiles, then there exist two ℓ -regular profiles, A_ℓ^* and B_ℓ^* , such that:*

1. $sc_f(W_1, A_\ell^*) = sc_f(W_3, A_\ell^*) > sc_f(W_2, A_\ell^*)$ and $W_1 \sim_{\mathcal{F}(A_\ell^*)} W_3 \succ_{\mathcal{F}(A_\ell^*)} W_2$,
2. $sc_f(W_1, B_\ell^*) = sc_f(W_3, B_\ell^*) < sc_f(W_2, B_\ell^*)$ and $W_1 \sim_{\mathcal{F}(B_\ell^*)} W_3 \prec_{\mathcal{F}(B_\ell^*)} W_2$.

Proof. Let c be a candidate such that $c \in W_1 \cap W_3$ and $c \notin W_2$. Such a candidate exists because $|W_1 \cap W_3| > |W_1 \cap W_2|$. Profile A_ℓ^* contains, for each $S \subseteq C \setminus \{c\}$ with $|S| = \ell - 1$, a vote with approval set $S \cup \{c\}$. First, let us note that all committees that contain c have the same f_ℓ -score in A_ℓ^* : this follows from neutrality, since the profile A_ℓ^* is symmetric with respect to committees containing c , in particular W_1 and W_3 . Let s denote the score of such committees.

Next, we will argue that $sc_{f_\ell}(W_2, A_\ell^*) < s$. To see this, let $c' \in W_2$ and consider a committee $W'_2 = (W_2 \setminus \{c'\}) \cup \{c\}$. Since f implements \mathcal{F} on ℓ -regular profiles, there exists $x \leq k$ such that $f_\ell(x, \ell) > f_\ell(x - 1, \ell)$. Due to Proposition 1 we can assume that $m - \ell \geq k - (x - 1)$; otherwise this difference between $f_\ell(x, \ell)$ and $f_\ell(x - 1, \ell)$ would not be relevant for computing scores. Let $T \subseteq C \setminus \{c, c'\}$ such that $|T| = \ell - 1$ and $|T \cap W_2| = x - 1$. To show that such a T exists, we have to prove that there exist $(\ell - 1) - (x - 1)$ candidates in $(C \setminus W_2) \setminus \{c, c'\}$. This is the case since $m - \ell \geq k - (x - 1)$ and thus $|(C \setminus W_2) \setminus \{c, c'\}| = m - k - 1 \geq \ell - x$.

Now let v be the vote in A_ℓ^* with approval set $T \cup \{c\}$. Since $f_\ell(x, \ell) > f_\ell(x - 1, \ell)$,

$$f_\ell(|A_\ell^*(v) \cap W'_2|, |A_\ell^*(v)|) > f_\ell(|A_\ell^*(v) \cap W_2|, |A_\ell^*(v)|).$$

Furthermore, for all votes v' in A_ℓ^* :

$$f_\ell(|A_\ell^*(v') \cap W'_2|, |A_\ell^*(v')|) \geq f_\ell(|A_\ell^*(v') \cap W_2|, |A_\ell^*(v')|).$$

Hence, $sc_{f_\ell}(W'_2, A_\ell^*) > sc_{f_\ell}(W_2, A_\ell^*)$. Since $f(x, \ell) = \gamma_\ell \cdot f_\ell(x, \ell)$ we get $sc_f(W'_2, A_\ell^*) > sc_f(W_2, A_\ell^*)$. Further, by a previous argument we have $sc_f(W_1, A_\ell^*) = sc_f(W'_2, A_\ell^*)$, thus by transitivity we conclude that $sc_f(W_1, A_\ell^*) > sc_f(W_2, A_\ell^*)$.

Next, let us construct profile B_ℓ^* . In this case we choose c such that $c \in W_2$ and $c \notin W_1 \cup W_3$. Again, this is possible because $|W_3 \setminus W_1| = k - |W_1 \cap W_3| < k - |W_1 \cap W_2| = |W_2 \setminus W_1|$ and hence $W_2 \not\subseteq W_1 \cup W_3$. Similarly as before, B_ℓ^* contains a vote with approval set $S \cup \{c\}$ for each $S \subseteq C \setminus \{c\}$ with $|S| = \ell - 1$. With similar arguments as before we can show that all committees that contain c have the same score in B_ℓ^* (in particular W_2) and this score is larger than the score of committees that do not contain c (in particular W_1 and W_3).

Finally, the statements concerning \mathcal{F} follow from Lemma 5 since both A_ℓ^* and B_ℓ^* are ℓ -regular. \square

We further generalize Lemma 7 and 8 so to allow us to compare W_1^* with arbitrary profiles (in particular, with profiles that have an arbitrary intersection with W_1^*). This is the final step; we can then proceed with a direct proof of Theorem 1.

Lemma 10. *For all $A \in \mathcal{A}(V)$ and $W \in \mathcal{P}_k(C)$ it holds that*

$$sc_f(W_1^*, A) > sc_f(W, A) \implies W_1^* \succ_{\mathcal{F}(A)} W.$$

Proof. We prove this statement by induction on ℓ -bounded profiles. As in Lemma 8, for 0-bounded profiles A the statement is trivial since $sc_f(W_1^*, A) > sc_f(W, A)$ cannot hold.

In order to prove the inductive step, we assume that the statement holds for ℓ -bounded profiles. Let A be an $(\ell + 1)$ -bounded profile and assume that $sc_f(W_1^*, A) > sc_f(W, A)$. We will show that $W_1^* \succ_{\mathcal{F}(A)} W$. If Case (A) or (B) was applicable when defining $\gamma_{\ell+1}$, the same arguments as in Lemma 8 yield that $W_1^* \succ_{\mathcal{F}(A)} W$.

If Case (C) was applicable when defining $\gamma_{\ell+1}$ and if $|W_1^* \cap W| = |W_1^* \cap W_2^*|$, then the statement of the lemma follows from Lemma 8 and neutrality. Recall that we fixed W_1^* and W_2^* as two committees with the smallest possible size of the intersection. Thus, if $|W_1^* \cap W| \neq |W_1^* \cap W_2^*|$ then $|W_1^* \cap W| > |W_1^* \cap W_2^*|$. For the sake of contradiction let us assume that $W \succeq_{\mathcal{F}(A)} W_1^*$. Let $sc_f(W_1^*, A) - sc_f(W, A) = \epsilon > 0$.

Now, from A we create a new profile B in the following way. Let us consider two cases:

Case 1: $sc_f(W_2^*, Bnd(A, \ell)) - sc_f(W, Bnd(A, \ell)) \geq 0$.

Let Q be an ℓ -bounded profile where:

$$sc_f(W_1^*, Q) = sc_f(W, Q) > sc_f(W_2^*, Q).$$

Such a profile exists due to Lemma 9. Since $sc_f(W_2^*, Q) - sc_f(W, Q)$ is negative, there exist such $x \in \mathbb{N}$, $y \in \mathbb{N} \cup \{0\}$ that $x \geq 2$ and

$$0 \leq \left(sc_f(W_2^*, Bnd(A, \ell)) - sc_f(W, Bnd(A, \ell)) \right) + y/x \cdot \left(sc_f(W_2^*, Q) - sc_f(W, Q) \right) < \epsilon/2,$$

which is equivalent to

$$0 \leq sc_f(W_2^*, xBnd(A, \ell) + yQ) - sc_f(W, xBnd(A, \ell) + yQ) < x\epsilon/2. \tag{A.6}$$

We set $B = xA + yQ$.

Case 2: $sc_f(W_2^*, Bnd(A, \ell)) - sc_f(W, Bnd(A, \ell)) < 0$.

In this case our reasoning is very similar. Let Q be an ℓ -bounded profile where:

$$sc_f(W_2^*, Q) > sc_f(W_1^*, Q) = sc_f(W, Q).$$

Again, similarly as before, we observe that there exist such $x, y \in \mathbb{N}$ that $x \geq 1$ and:

$$0 \leq \left(sc_f(W_2^*, Bnd(A, \ell)) - sc_f(W, Bnd(A, \ell)) \right) + y/x \cdot \left(sc_f(W_2^*, Q) - sc_f(W, Q) \right) < \epsilon/2,$$

which is equivalent to Inequality (A.6). Here, we also set $B = xA + yQ$.

By similar transformation as before, but applied to $Reg(B, \ell + 1)$ rather than to $Bnd(A, \ell)$, we construct a profile D from B :

Case 1: $sc_f(W_2^*, Reg(B, \ell + 1)) - sc_f(W, Reg(B, \ell + 1)) \geq 0$.

Due to Lemma 9 there exists an $(\ell + 1)$ -regular profile Q' with

$$sc_f(W_1^*, Q') = sc_f(W, Q') > sc_f(W_2^*, Q').$$

Similarly as before, there exist $x' \in \mathbb{N}$, $y' \in \mathbb{N} \cup \{0\}$ such that

$$0 \leq \text{sc}_f(W_2^*, x' \text{Reg}(B, \ell + 1) + y' Q') - \text{sc}_f(W, x' \text{Reg}(B, \ell + 1) + y' Q') < x' \epsilon / 2. \tag{A.7}$$

We set $D = x' B + y' Q'$.

Case 2: $\text{sc}_f(W_2^*, \text{Reg}(B, \ell + 1)) - \text{sc}_f(W, \text{Reg}(B, \ell + 1)) < 0$.

Here, let Q' be an $(\ell + 1)$ -regular profile such that

$$\text{sc}_f(W_1^*, Q') = \text{sc}_f(W, Q') > \text{sc}_f(W_2^*, Q').$$

There exist $x', y' \in \mathbb{N}$ such that Inequality (A.7) is satisfied. We set $D = x' B + y' Q'$.

Let us analyze the resulting profile $D = x' x A + x' y Q + y' Q'$. By our assumption we know that $W \succeq_{\mathcal{F}(A)} W_1^*$, thus by consistency we get that $W \succeq_{\mathcal{F}(xx'A)} W_1^*$. Since $W \sim_{\mathcal{F}(Q)} W_1^*$ and $W \sim_{\mathcal{F}(Q')} W_1^*$ due to Lemma 9, from consistency it follows that $W \succeq_{\mathcal{F}(D)} W_1^*$.

Further, since Q is ℓ -bounded and Q' is $(\ell + 1)$ -regular,

$$\begin{aligned} D &= x' x A + x' y Q + y' Q' \\ &= \text{Bnd}(x' x A + x' y Q + y' Q', \ell) + \text{Reg}(x' B + y' Q', \ell + 1) \\ &= \text{Bnd}(x' x A + x' y Q, \ell) + \text{Reg}(x' B + y' Q', \ell + 1) \\ &= x' \text{Bnd}(x A + y Q, \ell) + \text{Reg}(x' B + y' Q', \ell + 1). \end{aligned}$$

Inequalities (A.6) and (A.7) imply that W_2^* has a higher score than W in profiles $x'(x \text{Bnd}(A, \ell) + y Q) = x' \text{Bnd}(x A + y Q, \ell)$ and $x' \text{Reg}(B, \ell + 1) + y' Q' = \text{Reg}(x' B + y' Q', \ell + 1)$. From our inductive assumption we get that W_2^* is preferred over W in $x' \text{Bnd}(x A + y Q, \ell)$, and by Lemma 5 we get that W_2^* is preferred over W in $\text{Reg}(x' B + y' Q', \ell + 1)$. Consistency implies that $W_2^* \succeq_{\mathcal{F}(D)} W$, and thus $W_2^* \succeq_{\mathcal{F}(D)} W \succeq_{\mathcal{F}(D)} W_1^*$.

Now we observe that

$$\begin{aligned} &\text{sc}_f(W_1^*, \text{Bnd}(x A + y Q, \ell)) - \text{sc}_f(W_2^*, \text{Bnd}(x A + y Q, \ell)) \\ &= \left(\text{sc}_f(W_1^*, \text{Bnd}(x A + y Q, \ell)) - \text{sc}_f(W, \text{Bnd}(x A + y Q, \ell)) \right) \\ &\quad + \left(\text{sc}_f(W, \text{Bnd}(x A + y Q, \ell)) - \text{sc}_f(W_2^*, \text{Bnd}(x A + y Q, \ell)) \right) \\ &\geq \left(\text{sc}_f(W_1^*, \text{Bnd}(x A + y Q, \ell)) - \text{sc}_f(W, \text{Bnd}(x A + y Q, \ell)) \right) - \frac{x \epsilon}{2} \\ &= \left(\text{sc}_f(W_1^*, \text{Bnd}(x A, \ell)) - \text{sc}_f(W, \text{Bnd}(x A, \ell)) \right) - \frac{x \epsilon}{2}, \end{aligned}$$

and

$$\begin{aligned} &\text{sc}_f(W_1^*, \text{Reg}(x' B + y' Q', \ell + 1)) - \text{sc}_f(W_2^*, \text{Reg}(x' B + y' Q', \ell + 1)) \\ &= \left(\text{sc}_f(W_1^*, \text{Reg}(x' B + y' Q', \ell + 1)) - \text{sc}_f(W, \text{Reg}(x' B + y' Q', \ell + 1)) \right) \\ &\quad + \left(\text{sc}_f(W, \text{Reg}(x' B + y' Q', \ell + 1)) - \text{sc}_f(W_2^*, \text{Reg}(x' B + y' Q', \ell + 1)) \right) \\ &\geq \left(\text{sc}_f(W_1^*, \text{Reg}(x' B + y' Q', \ell + 1)) - \text{sc}_f(W, \text{Reg}(x' B + y' Q', \ell + 1)) \right) - \frac{x' \epsilon}{2} \\ &= \left(\text{sc}_f(W_1^*, \text{Reg}(x' B, \ell + 1)) - \text{sc}_f(W, \text{Reg}(x' B, \ell + 1)) \right) - \frac{x' \epsilon}{2} \\ &= \left(\text{sc}_f(W_1^*, \text{Reg}(x' x A, \ell + 1)) - \text{sc}_f(W, \text{Reg}(x' x A, \ell + 1)) \right) - \frac{x' \epsilon}{2}. \end{aligned}$$

By combining the above two inequalities we get that

$$\begin{aligned}
 &sc_f(W_1^*, D) - sc_f(W_2^*, D) \\
 &= x' \cdot \left(sc_f(W_1^*, Bnd(xA + yQ, \ell)) - sc_f(W, Bnd(xA + yQ, \ell)) \right) \\
 &\quad + \left(sc_f(W_1^*, Reg(x'B + y'Q', \ell + 1)) - sc_f(W, Reg(x'B + y'Q', \ell + 1)) \right) \\
 &\geq x' \cdot \left(sc_f(W_1^*, Bnd(xA, \ell)) - sc_f(W, Bnd(xA, \ell)) \right) \\
 &\quad + \left(sc_f(W_1^*, Reg(x'xA, \ell + 1)) - sc_f(W, Reg(x'xA, \ell + 1)) \right) - \frac{(x' + xx')\epsilon}{2} \\
 &= xx' \cdot \left(sc_f(W_1^*, A) - sc_f(W, A) \right) - \frac{(x' + xx')\epsilon}{2} \\
 &= xx'\epsilon - \frac{(x' + xx')\epsilon}{2} = \frac{(xx' - x')\epsilon}{2} > 0.
 \end{aligned}$$

Summarizing, we obtained a profile D , such that $sc_f(W_1^*, D) > sc_f(W_2^*, D)$ and $W_2^* \succ_{\mathcal{F}(D)} W_1^*$. This, however, contradicts Lemma 8. Hence, we have proven the inductive step, which completes the proof of the lemma. \square

Lemma 10 allows us to prove Theorem 1, our characterization of ABC scoring rules.

Finalizing the proof of Theorem 1. Let \mathcal{F} satisfy symmetry, consistency, weak efficiency, and continuity. If \mathcal{F} is trivial, then $f(x, y) = 0$ implements \mathcal{F} .

If \mathcal{F} is non-trivial, we construct f , W_1^* , and W_2^* as described above. We claim that for $A \in \mathcal{A}(V)$ and $W_1, W_2 \in \mathcal{P}_k(C)$ it holds that $sc_f(W_1, A) > sc_f(W_2, A)$ if and only if $W_1 \succ_{\mathcal{F}(A)} W_2$. For the “if” direction, fix W_1 and W_2 such that $sc_f(W_1, A) > sc_f(W_2, A)$ and consider a permutation $\sigma : C \rightarrow C$ such that $\sigma(W_1^*) = W_1$. Let $W = \sigma^{-1}(W_2)$. Since renaming the candidates does not change the scores of the renamed committees, $sc_f(W_1^*, \sigma^{-1}(A)) > sc_f(W, \sigma^{-1}(A))$ (here we renamed the candidates using σ^{-1}). By Lemma 10 we get that $W_1^* \succ_{\mathcal{F}(\sigma^{-1}(A))} W$. Applying neutrality with the permutation σ yields that $W_1 \succ_{\mathcal{F}(A)} W_2$.

Now, for the other direction, instead of showing that $W_1 \succ_{\mathcal{F}(A)} W_2$ implies $sc_f(W_1, A) > sc_f(W_2, A)$, we show that $sc_f(W_1, A) = sc_f(W_2, A)$ implies $W_1 \sim_{\mathcal{F}(A)} W_2$. Note that Lemma 10 does not apply to committees with the same score. For the sake of contradiction let $sc_f(W_1, A) = sc_f(W_2, A)$ but $W_1 \succ_{\mathcal{F}(A)} W_2$. As a first step, we prove that there exists a profile B with $sc_f(W_2, B) > sc_f(W_1, B)$ and $W_2 \succ_{\mathcal{F}(B)} W_1$. Since $W_1 \succ_{\mathcal{F}(A)} W_2$ and by neutrality, there exists a profile $A' \in \mathcal{A}(V)$ with $W_2 \succ_{\mathcal{F}(A')} W_1$. Thus, there exists an $\ell \in [m]$ such that $W_2 \succ_{\mathcal{F}(Reg(A', \ell))} W_1$, because otherwise, by consistency, $W_1 \succeq_{\mathcal{F}(A')} W_2$ would hold; let $B = Reg(A', \ell)$. Now, Lemma 5 guarantees that $sc_{f_\ell}(W_2, B) > sc_{f_\ell}(W_1, B)$. Since $f(x, \ell) = \gamma_\ell \cdot f_\ell(x, \ell)$, also $sc_f(W_2, B) > sc_f(W_1, B)$. Observe that for each $n \in \mathbb{N}$ we have $sc_f(W_2, B + nA) > sc_f(W_1, B + nA)$. Thus, by Lemma 10 for each n , $W_2 \succ_{\mathcal{F}(B+nA)} W_1$, which contradicts continuity of \mathcal{F} . Hence $sc_f(W_1, A) = sc_f(W_2, A)$ implies $W_1 \sim_{\mathcal{F}(A)} W_2$ and, consequently, $sc_f(W_1, A) > sc_f(W_2, A)$ if and only if $W_1 \succ_{\mathcal{F}(A)} W_2$. We see that f implements \mathcal{F} and thus \mathcal{F} is an ABC scoring rule.

Finally, as we already noted, an ABC scoring rule satisfies symmetry, consistency, weak efficiency, and continuity: this follows immediately from the definitions. \square

A.4. Independence of axioms

The set of axioms used in the statement of Theorem 1 is minimal. First, let us consider the variation of AV where the score of a fixed candidate c is doubled. Formally, the score of a committee W is defined as $\sum_{v \in V} |A(v) \cap W| + |\{v \in V : c \in A(v) \cap W\}|$. This rule satisfies all axioms except for neutrality. If we consider a variation of AV where voter 1 has a weight of 2, i.e., voter 1 gives a score of 2 to each approved candidate; all other voters have a weight of 1. This weighted AV rule clearly fails anonymity, but satisfies all other axioms. Note that here we need the fact that consistency only has to hold for disjoint voter sets (cf. footnote 3 on page 5).

Next, consider Proportional Approval Voting where ties are broken by Multi-Winner Approval Voting. This rule—let us call it \mathcal{F}^* —satisfies all axioms except for continuity: consider the profile $A = (\{c\})$ and $A' = (\{a, b\}, \{a, b\}, \{c\})$. It holds that $\{a, b\} \succ_{\mathcal{F}^*(A')} \{a, c\}$ because the PAV-score of both committees is 3, but the AV-score of $\{a, b\}$ is 4 and only 3 for $\{a, c\}$. However, it holds that $\{a, c\} \succ_{\mathcal{F}^*(A+nA')} \{a, b\}$ for arbitrary n because the PAV-scores of $\{a, c\}$ and $\{a, b\}$ are $3n + 1$ and $3n$, respectively.

To see that consistency is independent, consider an ABC ranking rule that is PAV on party-list profiles (i.e., D’Hondt) and the trivial rule otherwise. This rule fails consistency, since the addition of two party-list profiles may not be a party-list profile. All other axioms are satisfied by it: symmetry and weak efficiency are easy to see, continuity follows from the fact that in non-party-list profiles all committees are winning. Finally, the rule which reverses the output of Multi-Winner Approval Voting (i.e., $f(x, y) = -x$) satisfies all axioms except for weak efficiency.

Appendix B. Further proof details

Proposition 1. Let $D_{m,k} = \{(x, y) \in [0, k] \times [0, m - 1] : x \leq y \wedge k - x \leq m - y\}$ and let f, g be approval scoring functions. If there exist $c \in \mathbb{R}$ and $d : [m] \rightarrow \mathbb{R}$ such that $f(x, y) = c \cdot g(x, y) + d(y)$ for all $x, y \in D_{m,k}$ then f, g implement the same ABC scoring rule, i.e., for all approval profiles $A \in \mathcal{A}(V)$ and committees $W_1, W_2 \in \mathcal{P}_k(C)$ it holds that $sc_f(W_1, A) > sc_f(W_2, A)$ if and only if $sc_g(W_1, A) > sc_g(W_2, A)$.

Proof. Let $A \in \mathcal{A}(V)$ and $W \in \mathcal{P}_k(C)$. Let $D \subseteq [0, k] \times [0, m]$ be the domain of f and g that is actually used in the computation of $sc_f(W, A)$ and $sc_g(W, A)$. We will show that

$$D \subseteq D_{m,k} \cup \{(k, m)\}. \tag{B.1}$$

Let $v \in V$, $x = |A(v) \cap W|$, and $y = |A(v)|$. If $y = m$, then $x = |A(v) \cap W| = k$ and condition (B.1) is satisfied. Let $y < m$. If y is sufficiently large (close to m), then $A(v) \cap W$ cannot be empty. More precisely, it has to hold that the number of not approved members of W , $k - x$, is at most equal to the total number of not approved candidates in v , $m - y$; this yields that $k - x \leq m - y$. Furthermore, $x \leq y$ (the number of approved members of W must be at most equal to the total number of approved candidates). Consequently, $(x, y) \in D_{m,k}$. This shows that condition (B.1) holds.

Consider functions f and g as in the statement of the proposition. We will now show that for all $W_1, W_2 \in \mathcal{P}_k(C)$, it holds that:

$$sc_g(W_1, A) - sc_g(W_2, A) = c \cdot (sc_f(W_1, A) - sc_f(W_2, A)).$$

Let $V_i = \{v \in V : |A(v)| = i\}$ for $i \in [m]$. Now

$$\begin{aligned}
 & \text{sc}_g(W_1, A) - \text{sc}_g(W_2, A) = \\
 & = \sum_{i=1}^m \sum_{v \in V_i} g(|A(v) \cap W_1|, |A(v)|) - g(|A(v) \cap W_2|, |A(v)|) \\
 & = \sum_{i=1}^{m-1} \sum_{v \in V_i} \left(c \cdot f(|A(v) \cap W_1|, |A(v)|) + d(y) - c \cdot f(|A(v) \cap W_2|, |A(v)|) - d(y) \right) \\
 & = c \cdot \sum_{v \in V} \left(f(|A(v) \cap W_1|, |A(v)|) - f(|A(v) \cap W_2|, |A(v)|) \right) \\
 & = c \cdot (\text{sc}_f(W_1, A) - \text{sc}_f(W_2, A))
 \end{aligned}$$

Consequently, $\text{sc}_g(W_1, A) > \text{sc}_g(W_2, A)$ if and only if $\text{sc}_f(W_1, A) > \text{sc}_f(W_2, A)$. \square

Theorem 2. *Proportional Approval Voting is the only ABC scoring rule that satisfies D’Hondt proportionality.*

Proof. Theorem 2 is a special case of Theorem 5. \square

Theorem 3. *The Approval Chamberlin–Courant rule is the only non-trivial ABC scoring rule that satisfies disjoint diversity.*

Proof. The Approval Chamberlin–Courant rule maximizes the number of voters that have at least one approved candidate in the committee. In a party-list profile, this implies that the k largest parties receive at least one representative in the committee and hence disjoint diversity is satisfied.

For the other direction, let \mathcal{F} be an ABC scoring rule implemented by an approval scoring function f . Recall Proposition 1 and the relevant domain of approval scoring functions $D_{m,k} = \{(x, y) \in [0, k] \times [0, m - 1] : x \leq y \wedge k - x \leq m - y\}$. In a first step, we want to show that $f(x + 1, y) = f(x, y)$ for $x \geq 1$ and $(x + 1, y), (x, y) \in D_{m,k}$. Let us fix (x, y) such that $(x, y) \in D_{m,k}$, $(x + 1, y) \in D_{m,k}$, and $x \geq 1$. Furthermore, let us fix a committee W and consider a set $X \subseteq C$ with $|X| = y$ and $|X \cap W| = x$. We construct a party-list profile A as follows: A contains ζ votes that approve X (intuitively, ζ is a large natural number); further for each candidate $c \in W \setminus X$, profile A contains a single voter who approves $\{c\}$. This construction requires $y + (k - x)$ candidates. Since $(x, y) \in D_{m,k}$, we have $y + (k - x) \leq m$.

If we apply disjoint diversity to profile A , we obtain a winning committee W' with $W \setminus X \subseteq W'$ and $|W' \cap X| \geq 1$. Observe that $\text{sc}_f(W', A) = \text{sc}_f(W, A)$ (the satisfaction of all voters remains the same). Let W'' be the committee we obtain from W by replacing one candidate in $W \setminus X$ with a candidate in $X \setminus W$ (such a candidate exists since $(x + 1, y) \in D_{m,k}$). Since W is a winning committee, $\text{sc}_f(W'', A) \leq \text{sc}_f(W, A)$ and thus

$$\zeta f(x + 1, y) + (k - x - 1) f(1, 1) \leq \zeta f(x, y) + (k - x) f(1, 1). \tag{B.2}$$

The above condition can be written as $f(x + 1, y) - f(x, y) \leq \frac{1}{\zeta} \cdot f(1, 1)$. Since this must hold for any ζ , we get that $f(x + 1, y) \leq f(x, y)$. Since $f(x + 1, y) \geq f(x, y)$ by the definition of approval scoring functions, we get that $f(x + 1, y) = f(x, y)$ for $x \geq 1$. By Proposition 1 we can set $f(0, y) = 0$ for each $y \in [m]$. We conclude that \mathcal{F} is also implemented by the approval scoring function

$$f_\alpha(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ \alpha(y) & \text{if } x \geq 1. \end{cases}$$

As a next step we show that for the approval scoring function $f_\alpha(x, y)$ we can additionally assume that $\alpha(y) = \alpha(1)$, for each y . Observe that if $y \geq m - k + 1$, then for each committee W , a voter who approves y candidates in total, approves at least one member of W . By our previous reasoning, each committee gets from such a voter the same score, and so such a voter does not influence the outcome of an election. Consequently, we can assume that $\alpha(y) = \alpha(1)$ for $y > m - k$. Now, for $y \leq m - k$, we also show that $\alpha(y) = \alpha(1)$. Towards a contradiction assume that $\alpha(y) \neq \alpha(1)$ and further, without loss of generality, $\alpha(y) > \alpha(1)$. To this end, let n be natural number large enough so that $(n - 1) \cdot \alpha(y) > n \cdot \alpha(1)$. Consider a party-list profile consisting of $n - 1$ voters approving $\{c_1, \dots, c_y\}$, and, for $j \in [k]$, n voters each that approves candidate $\{c_{y+j}\}$. The committee $W_1 = \{c_{y+1}, \dots, c_{y+k}\}$ obtains a score of $nk \cdot f(1, 1) = nk \cdot \alpha(1)$, whereas $W_2 = \{c_1, c_{y+2}, \dots, c_{y+k}\}$ obtains a score of $(n - 1) \cdot \alpha(y) + n(k - 1) \cdot \alpha(1)$. Since by choice of n it holds that $(n - 1) \cdot \alpha(y) > n \cdot \alpha(1)$, committee W_2 is winning. This contradicts disjoint diversity and hence $\alpha(y) = \alpha(1)$.

Finally, we use Proposition 1 to argue that the CC scoring function f_{CC} implements \mathcal{F} . We distinguish two cases: $\alpha(1) > 0$ and $\alpha(1) = 0$. If $\alpha(1) > 0$, then $f_{CC} = \frac{1}{\alpha(1)} \cdot f_\alpha(x, y)$, and we see that Proposition 1 indeed applies.

If $\alpha(1) = 0$, then f_α is equivalent (by Proposition 1) to the trivial approval scoring function $f_0(x, y) = 0$. Since \mathcal{F} is non-trivial, this case cannot occur. \square

Theorem 4. *Multi-Winner Approval Voting is the only ABC scoring rule that satisfies disjoint equality.*

Proof. It is straightforward to verify that Multi-Winner Approval Voting satisfies disjoint equality. For the other direction, consider an ABC scoring rule satisfying disjoint equality that is implemented by an approval scoring function f . As in previous proofs we rely on Proposition 1 to show that f and $f_{AV}(x, y) = x$ implement the same ABC scoring rule. It is thus our aim to show that for $(x, y) \in D_{m,k}$ it holds that $f(x, y) = c \cdot x + d(y)$ for some $c \in \mathbb{R}$ and $d: [m] \rightarrow \mathbb{R}$. More specifically, we will show that for $(x, y) \in D_{m,k}$ with $0 \leq x < y$ it holds that $f(x + 1, y) - f(x, y) = f(1, 1) - f(0, 1)$. It then follows from induction that $f(x, y) = (f(1, 0) - f(0, 0)) \cdot x + f(0, y)$ and thus we will be able to conclude that f implements Multi-Winner Approval Voting.

Let $(x, y) \in D_{m,k}$ with $x < k$ and $x < y$. We construct a profile $A \in \mathcal{A}([k - x + 1])$ with $|A(1)| = y$ and $|A(2)| = \dots = |A(k - x + 1)| = 1$. All voters have disjoint sets of approved candidates. Hence this construction requires $y + k - x$ candidates. Since $(x, y) \in D_{m,k}$, it holds that $k - x \leq m - y$ and hence $y + k - x \leq m$; we see that a sufficient number of candidates is available. Let W_1 contain x candidates from $A(1)$ and one candidate from $A(2), \dots, A(k - x + 1)$ each. Let W_2 contain $x + 1$ candidates from $A(1)$ and one candidate from $A(2), \dots, A(k - x)$ each. Note that $|W_1| = |W_2| = k$. By disjoint equality both W_1 and W_2 are winning committees. Hence

$$f(x, y) + (k - x) \cdot f(1, 1) = f(x + 1, y) + (k - x - 1) \cdot f(1, 1) + f(0, 1)$$

and thus $f(x + 1, y) - f(x, y) = f(1, 1) - f(0, 1)$. \square

Theorem 5. Let $\mathbf{d} = (d_1, d_2, \dots)$ be a non-decreasing sequence of values from $\mathbb{N} \cup \{\infty\}$ and let $w = (1/d_1, 1/d_2, \dots)$. The w -Thiele method is the only ABC scoring rule that satisfies \mathbf{d} -proportionality.

Proof. To see that the w -Thiele method satisfies \mathbf{d} -proportionality, let f_{w-T} be the w -Thiele method's approval scoring function defined by $f_{w-T}(x, y) = \sum_{i=1}^x w_i$. Consider a party-list profile A with p parties, i.e., we have a partition of voters N_1, N_2, \dots, N_p and their corresponding joint approval sets C_1, \dots, C_p . For the sake of contradiction let us assume that $W \in \mathcal{P}_k(C)$ is a winning committee and that there exists i, j such that $\frac{|N_i|}{d_{|W \cap C_i|}} < \frac{|N_j|}{d_{|W \cap C_{j+1}|}}$, $W \cap C_i \neq \emptyset$ and $C_j \setminus W \neq \emptyset$. Let $a \in W \cap C_i$ and $b \in C_j \setminus W$. We define $W' = W \cup \{b\} \setminus \{a\}$. Let us compute the difference between w -scores of W and W' :

$$sc_{f_{w-T}}(W', A) - sc_{f_{w-T}}(W, A) = \frac{-|N_i|}{d_{|W \cap C_i|}} + \frac{|N_j|}{d_{|W \cap C_{j+1}|}} > 0.$$

Thus, we see that W' has a higher w -score than W , a contradiction.

To show the other direction, let \mathcal{F} be an ABC scoring rule that satisfies \mathbf{d} -proportionality and f its corresponding approval scoring function. We intend to apply Proposition 1 to show that f is equivalent to the w -Thiele method's approval scoring function $f_{w-T}(x, y) = \sum_{i=1}^x w_i$. Hence we have to show that there exists a constant c and a function $d: [m] \rightarrow \mathbb{R}$ such that $f(x) = c \cdot f_{w-T}(x, y) + d(y)$ for all $(x, y) \in D_{m,k} = \{(x, y) \in [0, k] \times [0, m - 1] : x \leq y \wedge k - x \leq m - y\}$.

Let us fix $x \in [k]$ such that $k - x < m - y$. We consider two cases: we start with the case when $d_x \neq \infty$.

d_x is a positive integer: Let us consider the following party-list profile. There are $k - x + 2$ groups of voters: N_1, \dots, N_{k-x+2} with $|N_1| = d_x$, $|N_i| = d_1$ for $i \geq 2$; their corresponding approval sets are C_1, \dots, C_{k-x+2} . Let $|C_1| = y$, $|C_i| = 1$ for $i \in [2, k - x + 1]$, and $|C_{k-x+2}| = m - y - k + x \geq 1$. Consider the two following committees: we choose W_1 such that $|W_1 \cap C_1| = x - 1$, $|W_1 \cap C_i| = 1$ for $i \geq 2$; we chose W_2 such that $|W_2 \cap C_1| = x$, $|W_2 \cap C_2| = 0$, and $|W_2 \cap C_i| = 1$ for $i \geq 1$.

It is straight-forward to verify that both W_1 and W_2 are \mathbf{d} -proportional.

Thus, W_1 and W_2 are winning committees and hence have the same scores. Their respective scores are

$$\begin{aligned} sc_f(W_1, A) &= d_x \cdot f(x - 1, y) + d_1 \cdot f(1, m - y - k + x) + (k - x) \cdot d_1 \cdot f(1, 1), \\ sc_f(W_2, A) &= d_x \cdot f(x, y) + d_1 \cdot f(1, m - y - k + x) + (k - x - 1) \cdot d_1 \cdot f(1, 1) \\ &\quad + d_1 \cdot f(0, 1). \end{aligned}$$

Since $sc_f(W_1, A) = sc_f(W_2, A)$ we have

$$f(x, y) = f(x - 1, y) + \frac{d_1}{d_x} \left(f(1, 1) - f(0, 1) \right).$$

$d_x = \infty$: Now, let us move to the case when $d_x = \infty$. Let us fix a committee W and consider a set $X \subseteq C$ with $|X| = y$ and $|X \cap W| = x - 1$. We construct a party-list profile A as follows: A contains ζ votes that approve X (intuitively, ζ is a large natural number); further for each candidate $c \in W \setminus X$, profile A contains a single voter who approves $\{c\}$. This construction requires $y + (k - x + 1)$ candidates, thus it is possible since we fixed x so that $k - x < m - y$.

Clearly, committee W is \mathbf{d} -proportional. Let W' be the committee we obtain from W by replacing one candidate in $W \setminus X$ with a candidate in $X \setminus W$ (such a candidate exists since $(x, y) \in D_{m,k}$). We have $sc_f(W', A) \leq sc_f(W, A)$ and thus

$$\zeta f(x, y) + (k - x)f(1, 1) + f(0, 1) < \zeta f(x - 1, y) + (k - x + 1)f(1, 1). \tag{B.3}$$

The above condition can be written as $f(x, y) - f(x - 1, y) \leq \frac{1}{\zeta} \cdot (f(1, 1) - f(0, 1))$. Since this must hold for any ζ , we get that $f(x + 1, y) \leq f(x, y)$. Since f is an approval scoring function, $f(x, y) \geq f(x - 1, y)$; thus we get that $f(x, y) = f(x - 1, y)$, i.e.:

$$f(x, y) = f(x - 1, y) + \frac{d_1}{\infty} (f(1, 1) - f(0, 1)).$$

(Above, we use the convention that $\frac{\infty}{\infty} = 0$.)

Now, as we have shown that

$$f(x, y) = f(x - 1, y) + \frac{d_1}{dx} (f(1, 1) - f(0, 1))$$

holds for $1 \leq x \leq k$ such that $k - x < m - y$, we can expand this equation until we reach $x = 0$ or $x = k + y - m$. Let $s(y) = \max(0, k + y - m)$.

$$\begin{aligned} f(x, y) &= f(s(y), y) + d_1 (f(1, 1) - f(0, 1)) \sum_{i=s(y)+1}^x w_i \\ &= f(s(y), y) - d_1 (f(1, 1) - f(0, 1)) \sum_{i=1}^{s(y)} w_i + d_1 (f(1, 1) - f(0, 1)) \sum_{i=1}^x w_i. \end{aligned}$$

Obviously, the above equality also holds for $x = s(y)$.

Hence we have shown that indeed $f(x) = c \cdot f_{w-T}(x, y) + d(y)$ for $c = d_1(f(1, 1) - f(0, 1))$ and $d(y) = f(s(y), y) - d_1 (f(1, 1) - f(0, 1)) \sum_{i=1}^{s(y)} w_i$. By Proposition 1, \mathcal{F} is w -Thiele. \square

Lemma 1. *Let $d = (d_1, d_2, \dots)$ be a non-decreasing sequence of values from \mathbb{N} . An ABC ranking rule that satisfies neutrality, consistency, and \mathbf{d} -proportionality also satisfies weak efficiency.*

Proof. Let \mathcal{F} be an ABC ranking rule satisfying symmetry, consistency, and \mathbf{d} -proportionality. To show that \mathcal{F} satisfies weak efficiency, it suffices to show that \mathcal{F} satisfies weak efficiency for single-voter profiles. Indeed, assume that \mathcal{F} satisfies weak efficiency for single-voter profiles. Let $W_1, W_2 \in \mathcal{P}_k(C)$ and $A \in \mathcal{A}(V)$ where no voter approves a candidate in $W_2 \setminus W_1$; we want to show that $W_1 \succeq_{\mathcal{F}(A)} W_2$. Since weak efficiency holds for single-voter profiles, we know that $W_1 \succeq_{\mathcal{F}(A(v))} W_2$ for all $v \in V$. By consistency we can infer that $W_1 \succeq_{\mathcal{F}(A)} W_2$.

For the sake of contradiction let us assume that \mathcal{F} does not satisfy weak efficiency for single-voter profiles. This means that there exist $X \subseteq C$ and $W_1, W_2 \in \mathcal{P}_k(C)$ such that $(W_2 \setminus W_1) \cap X = \emptyset$ and $W_2 \succ_{\mathcal{F}(X)} W_1$. First, we show that in such case there exist $W \in \mathcal{P}_{k-1}(C)$, $c, c' \in C$ with $c \in X$, $c' \notin X$, and $W \cup \{c'\} \succ_{\mathcal{F}(X)} W \cup \{c\}$. Let $z = |W_1 \cap X| - |W_2 \cap X|$, and let us consider the following sequence of z operations which define z new committees. We start with committee $W_{2,1} = W_2$, and in the i -th operation, $i \in [z - 1]$, we construct $W_{2,i+1}$ from $W_{2,i}$ by removing from $W_{2,i}$ one arbitrary candidate in $W_{2,i} \cap X$ and by adding one candidate from $(W_1 \setminus W_2) \cap X$. Consequently, $|W_{2,z} \cap X| = |W_1 \cap X|$, so by neutrality we have $W_{2,z} \sim_{\mathcal{F}(X)}$

W_1 . By our assumption we have that $W_{2,1} \succ_{\mathcal{F}(X)} W_{2,z}$, thus, there exists $i \in [z - 1]$ such that $W_{2,i} \succ_{\mathcal{F}(X)} W_{2,i+1}$. The committees $W_{2,i}$ and $W_{2,i+1}$ differ by one element only, so we set $W = W_{2,i} \cap W_{2,i+1}$, $c \in W_{2,i+1} \setminus W_{2,i}$ and $c' \in W_{2,i} \setminus W_{2,i+1}$, and we have $W \cup \{c'\} \succ_{\mathcal{F}(X)} W \cup \{c\}$ for $c \in X$ and $c' \notin X$.

Let ℓ denote the number of members of $W \cup \{c\}$ which are approved in X , i.e., $\ell = |(W \cup \{c\}) \cap X|$. Let us consider the following party-list profile A' . There are two groups of voters: N_1 with $|N_1| = d_\ell$ and N_2 with $|N_2| = d_{k-\ell}$. The voters in N_1 approve of X ; the voters in N_2 approve $C \setminus (X \cup \{c'\})$. From \mathbf{d} -proportionality we infer that committee $W \cup \{c\}$ is winning:

$$\frac{|N_1|}{d_{|(W \cup \{c\}) \cap X|}} = 1 \geq \frac{d_{k-\ell}}{d_{k-\ell+1}} = \frac{|N_2|}{d_{|(W \cup \{c\}) \cap (C \setminus (X \cup \{c'\}))|+1}},$$

$$\frac{|N_2|}{d_{|(W \cup \{c\}) \cap (C \setminus (X \cup \{c'\}))|}} = 1 \geq \frac{d_\ell}{d_{\ell+1}} = \frac{|N_1|}{d_{|(W \cup \{c\}) \cap X|+1}}.$$

This, however, yields a contradiction: Voters from N_1 prefer $W \cup \{c'\}$ over $W \cup \{c\}$ since $W \cup \{c'\} \succ_{\mathcal{F}(X)} W \cup \{c\}$. For voters from N_2 committees $W \cup \{c'\}$ and $W \cup \{c\}$ are equally good by neutrality. Hence, by consistency, it holds that $W \cup \{c'\} \succ_{\mathcal{F}(A')} W \cup \{c\}$, a contradiction. We conclude that $W_1 \succeq_{\mathcal{F}(X)} W_2$ and hence weak efficiency holds for single-voter profiles and—in consequence—for arbitrary profiles. \square

Proposition 2. Fix $x, y \in \mathbb{N}$ and let $m \geq y + k - x + 1$. Let \mathcal{F} be an ABC scoring rule satisfying lower quota, and let f be an approval scoring function implementing \mathcal{F} . It holds that:

$$f(x - 1, y) + \frac{1}{x} \cdot f(1, 1) \cdot \frac{k - x}{k - x + 1} \leq f(x, y) \leq f(x - 1, y) + \frac{1}{x - 1} \cdot f(1, 1).$$

Proof. Consider a party-list profile A with one group of voters N_1 approving y candidates and $k - x + 1$ groups of voters, N_2, \dots, N_{k-x+2} , each approving a single candidate—for each $i \in [k - x + 2]$ let C_i denote the set of candidates approved by voters from N_i . Each of the remaining $m - y - k + x - 1$ candidates is not approved by any voter. We set $|N_1| = x(k - x + 1)$, and for each $i \geq 2$ we set $|N_i| = k - x$. Observe that:

$$k \cdot \frac{|N_1|}{|V|} = k \cdot \frac{x(k - x + 1)}{x(k - x + 1) + (k - x + 1)(k - x)} = k \cdot \frac{x(k - x + 1)}{k(k - x + 1)} = x.$$

From the lower quota property, we infer that there exists a winning committee W such that $|W \cap C_1| \geq x$, and from the pigeonhole principle we get that there exists $i \geq 2$ with $W \cap C_i = \emptyset$; let $C_i = \{c_i\}$. Thus, the score of committee W is higher than or equal to the score of committee $(W \cup \{c_i\}) \setminus \{c\}$ for $c \in W \cap C_1$. As a result we get that $f(x, y)|N_1| \geq f(x - 1, y)|N_1| + f(1, 1)|N_i|$, which can be equivalently written as:

$$f(x, y) \geq f(x - 1, y) + \frac{1}{x} \cdot f(1, 1) \cdot \frac{k - x}{k - x + 1}.$$

Now, consider another similar party-list profile, with the only difference that $|N_1| = x - 1$, and $|N_i| = 1$ for $i \geq 2$. Observe that for $i \geq 2$:

$$k \cdot \frac{|N_i|}{|V|} = k \cdot \frac{1}{x - 1 + (k - x + 1)} = 1.$$

Thus, for each $i \geq 2$ we have that $|W \cap C_i| = 1$. By a similar reasoning as before we get that: $f(1, 1)|N_i| + f(x - 1, y)|N_1| \geq f(x, y)|N_1|$, which is equivalent to:

$$f(x, y) \leq f(x - 1, y) + \frac{1}{x - 1} \cdot f(1, 1).$$

This completes the proof. \square

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