# The Complexity of Nearly Single-Peaked Consistency[1]

Gábor Erdélyi, Martin Lackner, and Andreas Pfandler

**Abstract**

Manipulation, bribery, and control are well-studied ways of changing the outcome of an election. Many voting systems are in the general case computationally resistant to some of these manipulative actions. However when restricted to single-peaked electorates, these problems suddenly become easy to solve. Recently, Faliszewski, Hemaspaandra, and Hemaspaandra [FHH11] studied the complexity of dishonest behavior in nearly single-peaked electorates. These are electorates that are not single-peaked but close to it according to some distance measure.

In this paper we introduce several new distance measures regarding single-peakedness. We prove that determining whether a given profile is nearly single-peaked is in many cases NP-complete. Furthermore, we explore the relations between several notions of nearly single-peakedness.

## 1 Introduction

Voting is a very useful method for preference aggregation and collective decision-making. It has applications in very broad settings ranging from politics to artificial intelligence and further topics in computer science (see, e.g., [DKNS01, ER97, GMHS99]). In the presence of huge data volumes, the computational properties of voting rules are worth studying. In particular, we usually want to determine the winners of an election quickly. On the other hand we want to make various forms of dishonest behavior computationally as hard as possible.

Bartholdi, Tovey, and Trick [BTT89] were the first to study the computational aspects of manipulation in elections, where a group of voters cast their votes insincerely in order to reach a desired outcome. Other types of dishonest behavior are control, where an external agent makes structural changes on the election such as adding/deleting/partitioning either candidates or voters (see, e.g., [BTT92]) in order to reach a desired outcome, or bribery, where an external agent changes some voters' votes in order to change the outcome of the election (see, e.g., [FHH09]). For an overview and many natural examples on bribery, control, and manipulation we refer to the survey of Baumeister et al. [BEH$^+$10].

Traditionally, the complexity of such attacks on the outcome is studied under the assumption that in each election any admissible vote can occur. However, there are many elections where the diversity of the votes is limited in a sense that there are some admissible votes nobody would ever cast. One of the best known examples is *single-peakedness*, introduced by Black [Bla48]. It assumes that the votes are polarized along some linear axis. The study of the computational aspects of elections with single-peaked preferences was initiated by Walsh [Wal07] (see also [FHHR11, BBHH10]). In many cases NP-hardness results from the general cases turn out to be easy in single-peaked societies. A recent line of research initiated by Conitzer [Con09] and by Escoffier, Lang, and Öztürk [ELÖ08] suggests that many elections are not perfectly single-peaked but are *close* to it with respect to some metric. Faliszewski, Hemaspaandra, and Hemaspaandra [FHH11] introduced various notions of nearly single-peaked elections and showed that the complexity of manipulative-actions jumps back to NP-hardness in many cases.

In this paper we consider the notion of $k$-maverick single-peakedness and $k$-local swaps introduced by Faliszewski, Hemaspaandra, and Hemaspaandra [FHH11]. In addition we follow the sug-

[1]This work was done in part while the second and the third authors were visiting Universität Siegen and while the first author was visiting Vienna University of Technology. The work of the second and third author was supported by the Austrian Science Fund (FWF): P20704-N18.

gestions of Escoffier, Lang, and Öztürk [ELÖ08] and formally define the two nearly single-peaked notions $k$-candidate deletion and $k$-additional axes. Furthermore, we introduce three new notions of nearly single-peakedness, $k$-local candidate deletion, $k$-global swaps, and $k$-candidate partition. We show connections between the existing and new notions, and we study the complexity of determining whether a given profile is nearly single-peaked with respect to some axis. This problem was introduced by Escoffier, Lang, and Öztürk [ELÖ08] as *single-peaked consistency*. We show that single-peaked consistency is computationally hard for four notions of nearly single-peakedness given in this paper. The complexity of the remaining three notions is still open.

**Related Work** Our paper fits in the line of research on single-peaked and nearly single-peaked preferences. Faliszewski et al. [FHHR11] and Brandt et al. [BBHH10] investigate the complexity of dishonest behavior (e.g., the complexity of manipulation and control) in electorates with single-peaked preferences as well as the winner problem. They do not consider nearly single-peaked preferences, but mention them as future work.

In the context of nearly single-peaked preferences the most relevant paper is by Faliszewski, Hemaspaandra, and Hemaspaandra [FHH11]. They introduce several notions of nearly single-peakedness and analyze the complexity of bribery, control, and manipulation under those conditions. In contrast, we are not analyzing dishonest behavior in elections, but we are studying the complexity of nearly single-peaked consistency.

The question whether a given profile is single-peaked has been recently investigated by Escoffier, Lang, and Öztürk [ELÖ08]. The difference in their work is that they have not considered nearly single-peakedness but they only pointed it out as a possible future research direction.

The idea of measuring the distance of votes with the number of required swaps required to make them identical already appears in Dodgon's voting rule (see, e.g., [MN08] for a discussion). This idea has been widely used since then. Elkind, Faliszewski, and Slinko used swaps of adjacent candidates in votes in the context of bribery [EFS09]. They assumed that a briber can perform a number of swaps in the votes in order to make his favourite candidate win the election. In our paper, we use swaps as a distance measure for nearly single-peakedness. We do not want to change the outcome of an election, we just want to measure the swap distance of a given profile to the nearest single-peaked profile.

Finally, we remark that single-peaked preferences have been considered in the context of preference elicitation [Con09] and in the context of possible and necessary winners under uncertainty regarding the votes [Wal07].

**Organization** This paper is organized as follows. In Section 2, we recall some notions from voting theory and define single-peaked preferences. In Section 3, we introduce the problems we are investigating in our paper. Our results on the relations between the different notions of single-peakedness and on the complexity of single-peaked consistency are presented in Section 4. Finally, Section 5 provides some conclusions and future directions.

## 2 Preliminaries

Let $C$ be a finite set of *candidates*, $V$ be a finite set of *voters*, and let $\succ$ be a *preference relation* (i.e., a tie-free and total order) over $C$. We call a candidate $c$ the *peak* of a preference relation $\succ$ if $c \succ c_i$ for all $c_i \in C \setminus \{c\}$. Let $\mathscr{P} = (\succ_1, \ldots, \succ_n)$ be a *preference profile* (i.e., a collection of linear orders) over the candidate set $C$. We say that the preference order $\succ_i$ is the *vote* of voter $i$. For simplicity, we will write for each voter $i \in V$ $c_1 c_2 \ldots c_n$ instead of $c_1 \succ_i c_2 \succ_i \ldots \succ_i c_n$. We call the peak of voter $i$ his highest ranked or *top-ranked* candidate. An *election* is defined as a triple $E = (C, V, \mathscr{P})$, where $C$ is the set of candidates, $V$ the set of voters and $\mathscr{P}$ a preference profile over $C$.

In order to define single-peaked profiles we will make use of the definition given by Escoffier et al. [ELÖ08].

**Definition 2.1** ([ELÖ08]). *Let an* axis *$A$ be a total order over $C$ denoted by $>$. Given two candidates $c_i, c_j \in C$, a vote $k \in V$ specified by the corresponding preference relation $\succ_k$, and an axis $A$. Let $c$ be the top-ranked candidate of voter $k$. We say that candidates $c_i$ and $c_j$ are* on the same side of the peak of $\succ_k$ *if one of the following two conditions holds:*

$$(1) \quad c_i > c \text{ and } c_j > c, \text{ or} \qquad\qquad (2) \quad c > c_i \text{ and } c > c_j$$

*A vote $k$ is said to be* single-peaked with respect to an axis $A$ *if for all $c_i, c_j \in C$ that are on the same side of the peak $c$ of $\succ_k$ it holds that $c_i \succ_k c_j$ if either $c > c_i > c_j$ or $c_j > c_i > c$ holds (i.e., $c_i$ is closer to the peak than $c_j$).*

*A preference profile $\mathscr{P}$ is said to be* single-peaked with respect to an axis $A$ *if and only if each vote is single-peaked with respect to $A$. A preference profile $\mathscr{P}$ is said to be* single-peaked consistent *if there is an axis $A$ such that $\mathscr{P}$ is single-peaked with respect to $A$.*

Let $C' \subseteq C$. By $\mathscr{P}[C']$ we denote the profile $\mathscr{P}$ restricted to the candidates in $C'$. Analogously if $A$ is an axis over $C$, we denote by $A[C']$ the axis $A$ restricted to candidates in $C'$.

Escoffier, Lang, and Öztürk present an algorithm that decides whether a given preference profile is single-peaked consistent in time $|V| \cdot |C|$ [ELÖ08]. Their algorithm improves upon the runtime of an algorithm presented in [BT86]. The corresponding decision problem is defined as follows.

| Single-Peaked Consistency | |
|---|---|
| **Given:** | An election $E = (C, V, \mathscr{P})$. |
| **Question:** | Is $\mathscr{P}$ single-peaked consistent? |

# 3 Problem Statement

In this paper we consider different notions of nearly single-peakedness. All these notions define a distance measure to single-peaked profiles. We will now describe them and provide first (trivial) upper bounds on these distances.

### *k-Maverick*

The first formal definition of nearly single-peaked societies was given by Faliszewski, Hemaspaandra, and Hemaspaandra [FHH11]. Consider a preference profile $\mathscr{P}$ for which most voters are single-peaked with respect to some axis $A$. All voters that are not single-peaked with respect to $A$ are called mavericks. The number of mavericks defines a natural distance measure to single-peakedness. If an axis can be found for a large subset of the voters, this is still a fundamental observation about the structure of the votes.

**Definition 3.1** ([FHH11]). *Let $E = (C, V, \mathscr{P})$ be an election and $k$ a positive integer. We say that the profile $\mathscr{P}$ is* $k$-maverick single-peaked consistent *if by removing at most $k$ preference relations (votes) from $\mathscr{P}$ one can obtain a preference profile $\mathscr{P}'$ that is single-peaked consistent.*

Let $M(\mathscr{P})$ denote the smallest $k$ such that $\mathscr{P}$ is $k$-maverick single-peaked consistent. Note that $M(\mathscr{P}) \leq |V| - 1$ always holds.

The above notion is a well-motivated distance regarding single-peakedness, but we will define other distances which could be more useful in other cases.

### *k*-Candidate Deletion

As suggested in [ELÖ08], we introduce outlier candidates. These are candidates that do not have "a correct place" on any axis and consequently have to be deleted in order to obtain a single-peaked consistent profile. Examples could be a candidate that is not well-known (e.g., a new political party) or a candidate that prioritizes other topics than most candidates and thereby is judged by the voters according to different criteria. The votes restricted to the remaining candidates might still have a clear and significant structure, i.e., might be single-peaked consistent.

**Definition 3.2.** *Let $E = (C, V, \mathscr{P})$ be an election and $k$ a positive integer. We say that the profile $\mathscr{P}$ is $k$-*candidate deletion single-peaked consistent *if we can obtain a set $C' \subseteq C$ by removing at most $k$ candidates from $C$ such that the preference profile $\mathscr{P}[C']$ is single-peaked consistent.*

Let $CD(\mathscr{P})$ denote the smallest $k$ such that $\mathscr{P}$ is $k$-candidate deletion single-peaked consistent. Note that $CD(\mathscr{P}) \leq |C| - 2$ always holds.

### *k*-Local Candidate Deletion

Personal friendships or hatreds between voters and candidates could move candidates up or down in a vote. These personal relationships cannot be reflected in a global axis. To eliminate the influence of personal relationships to some candidates we define a local version of the previous notion. This notion can also deal with the possibility that the least favourite candidates might be ranked without special consideration or even randomly.

We first have to define partial domains and partial profiles.

**Definition 3.3.** *Let $C$ be a set of candidates and $A$ an axis over $C$. A preference relation $\succ$ over a candidate set $C' \subset C$ is called a partial vote. It is said to be* single-peaked with respect to $A$ *if it is single-peaked with respect to $A[C']$. A partial preference profile consists of partial votes. It is called* single-peaked consistent *if there exists an axis $A$ such that its partial votes are single-peaked with respect to $A$.*

**Definition 3.4.** *Let $E = (C, V, \mathscr{P})$ be an election and $k$ a positive integer. We say that the profile $\mathscr{P}$ is $k$-*local candidate deletion single-peaked consistent *if by removing at most $k$ candidates from each vote in $V$ we obtain a partial preference profile $\mathscr{P}'$ that is single-peaked consistent.*

Let $LCD(\mathscr{P})$ denote the smallest $k$ such that $\mathscr{P}$ is $k$-local candidate deletion single-peaked consistent. Note that $LCD(\mathscr{P}) \leq |C| - 2$ always holds.

### *k*-Additional Axes

Another suggestion in [ELÖ08] is to consider the minimum number of axes such that each preference relation of the profile is single-peaked with respect to at least one of these axes. This notion is particularly useful if each candidate represents opinions on several issues (as it is the case in political elections). A voter's ranking of the candidates would then depend on which issue is considered most important by the voter and consequently each issue might give rise to its own corresponding axis.

**Definition 3.5.** *Let $E = (C, V, \mathscr{P})$ be an election and $k$ a positive integer. We say that the profile $\mathscr{P}$ is $k$-*additional axes single-peaked consistent *if there is a partition $V_1, \dots, V_{k+1}$ of $V$ such that the corresponding preference profiles $\mathscr{P}_1, \dots, \mathscr{P}_{k+1}$ are single-peaked consistent.*

Let $AA(\mathscr{P})$ denote the smallest $k$ such that $\mathscr{P}$ is $k$-additional axes single-peaked consistent. Note that $AA(\mathscr{P}) < \min\left(|V|, \frac{|C|!}{2}\right)$ always holds. This is because the number of distinct votes is trivially bounded by $|V|$. Furthermore, $AA(\mathscr{P})$ is bounded by $\frac{|C|!}{2}$ since at most $|C|!$ distinct votes exist and each vote and its reverse are single-peaked with respect to the same axes.

### *k*-Global Swaps

There is a second method of dealing with candidates that are "not placed correctly" according to an axis $A$. Instead of deleting them from either the candidate set $C$ or from a vote, we could try to move them to the right position. We do this by performing a sequence of swaps of consecutive candidates. For example, to get from vote *abcd* to vote *adbc*, we first have to swap candidates $c$ and $d$, and then we have to swap $b$ and $d$. Since this changes the votes in a more subtle way, this can be considered a less obtrusive notion than $k$-(Local) Candidate Deletion.

**Definition 3.6.** *Let $E = (C, V, \mathscr{P})$ be an election and k a positive integer. We say that the profile $\mathscr{P}$ is k-*global swaps single-peaked consistent *if $\mathscr{P}$ can be made single-peaked by performing at most k swaps in the profile. (Note that these swaps can be performed wherever we want – we can have k swaps in only one vote, or one swap each in k votes.)*

Let $GS(\mathscr{P})$ denote the smallest $k$ such that $\mathscr{P}$ is $k$-global swaps single-peaked consistent. Note that $GS(\mathscr{P}) \leq \binom{|C|}{2} \cdot |V|$ always holds since rearranging a total order in order to obtain any other total order requires at most $\binom{|C|}{2}$ swaps.

### *k*-Local Swaps

We can also consider a "local budget" for swaps, i.e., we allow up to $k$ swaps per vote. This distance measure has been introduced in [FHH11] as Dodgson$_k$.

**Definition 3.7.** *Let $E = (C, V, \mathscr{P})$ be an election and k a positive integer. We say that the profile $\mathscr{P}$ is k-*local swaps single-peaked consistent *if $\mathscr{P}$ can be made single-peaked consistent by performing no more than k swaps per vote.*

Let $LS(\mathscr{P})$ denote the smallest $k$ such that $\mathscr{P}$ is $k$-local swaps single-peaked consistent. Note that $LS(\mathscr{P}) \leq \binom{|C|}{2}$ always holds.

### *k*-Candidate Partition

Our last nearly single-peaked formalism is the candidate analogon of $k$-additional axes. In this case we partition the set of candidates into subsets such that all of the restricted profiles are single-peaked consistent. This notion is useful in the following situation. Each candidate has an opinion on a controversial Yes/No-issue. Depending on their own preference voters will always rank all Yes-candidates before or after all No-candidates. It might be that when considering only the Yes- respectively No-candidates, the election is single-peaked. Therefore, if we acknowledge the importance of this Yes/No-issue and partition the candidates accordingly, we may obtain two single-peaked elections.

**Definition 3.8.** *Let $E = (C, V, \mathscr{P})$ be an election and k a positive integer. We say that the profile $\mathscr{P}$ is k-*candidate partition single-peaked consistent *if the set of candidates C can be partitioned into at most k disjoint sets $C_1, \dots, C_k$ with $C_1 \cup \dots \cup C_k = C$ such that the profiles $\mathscr{P}[C_1], \dots, \mathscr{P}[C_k]$ are single-peaked consistent.*

Let $CP(\mathscr{P})$ denote the smallest $k$ such that $\mathscr{P}$ is $k$-candidate partition single-peaked consistent. Note that $CP(\mathscr{P}) \leq \left\lceil \frac{|C|}{2} \right\rceil$ always holds.

### Decision Problems

We now introduce the seven problems we will study. We define the following problem for X $\in$ {Maverick, Candidate Deletion, Local Candidate Deletion, Additional Axes, Global Swaps, Local Swaps, Candidate Partition}.

| X SINGLE-PEAKED CONSISTENCY | |
|---|---|
| **Given:** | An election $E = (C, V, \mathscr{P})$ and a positive integer $k$. |
| **Question:** | Is $\mathscr{P}$ $k$-X single-peaked consistent? |

# 4 Results

## 4.1 Basic Results about Single-Peaked Profiles

We start with a simple observation which we will use in the proof of Theorem 4.6.

**Lemma 4.1.** *Let $\mathscr{P}$ be a preference profile containing the preference relation $\succ_1$: $c_1 \dots c_n$ and its reverse $\succ_2$: $c_n \dots c_1$. Then $\mathscr{P}$ is either single-peaked with respect to the axis $c_1 < \dots < c_n$ (and its reverse) or it is not single-peaked at all.*

**Proof.** Since the vote $\succ_1$ ranks $c_n$ last while the vote $\succ_2$ ranks $c_1$ last, these candidates have to be at the left-most and right-most position on any compatible axis. Note that $c_1$ is the peak in $\succ_1$. Hence this already determines the position of all other candidates. Consequently only two axes are possible: $c_1 < \dots < c_n$ and $c_n < \dots < c_1$. Since any preference profile is single-peaked with respect to $c_1 < \dots < c_n$ if and only if it is single-peaked with respect to $c_n < \ldots < c_1$, we can focus without loss of generality on the former. ❑

Lemma 4.2 provides an alternative characterization of single-peaked consistency.

**Lemma 4.2.** *Given an election $(C, V, \mathscr{P})$, the profile $\mathscr{P}$ is not single-peaked consistent if and only if for all axes A there is some voter $v \in V$ and three candidates $c_i, c_j, c_k \in C$ such that $c_i > c_j > c_k$ on axis A, and $c_i \succ_v c_j$ holds as well as $c_k \succ_v c_j$.*

The following observation says that any subelection, i.e., an election with the same voters over a subset of the candidate set, of a single-peaked election is also single-peaked.

**Lemma 4.3.** *Let $(C, V, \mathscr{P})$ be a given election and $C' \subseteq C$. If $\mathscr{P}$ is single-peaked consistent then also $\mathscr{P}[C']$ is single-peaked consistent.*

In the constructions in our main results we will have to cascade two or more preference profiles. The following definition captures this notion.

**Definition 4.4.** *Let $(C_1, V, \mathscr{P}_1)$ and $(C_2, V, \mathscr{P}_2)$ be two elections with $C_1 \cap C_2 = \emptyset$. Furthermore, let $\mathscr{P}_1 = (\succ'_1, \dots, \succ'_n)$ and $\mathscr{P}_2 = (\succ''_1, \dots, \succ''_n)$. We define $\mathscr{P}_1 \circledgt \mathscr{P}_2 = (\succ_1, \dots, \succ_n)$, where for any $1 \leq i \leq n$ the linear order $\succ_i$ is defined by*

$$c \succ_i c' \text{ iff } (c, c' \in C_1 \text{ and } c \succ'_i c') \text{ or } (c, c' \in C_2 \text{ and } c \succ''_i c') \text{ or } (c \in C_1 \text{ and } c' \in C_2).$$

Note that $\mathscr{P}_1 \circledgt \mathscr{P}_2$ is always a preference profile over $C_1 \cup C_2$.

**Lemma 4.5.** *Let $(C_1, V, \mathscr{P}_1)$ and $(C_2, V, \mathscr{P}_2)$ be two elections with $C_1 \cap C_2 = \emptyset$. Assume that*

- *$\mathscr{P}_1$ and $\mathscr{P}_2$ are single-peaked consistent with respect to the axes $A_1$ and $A_2$, respectively.*
- *The preference relations in $\mathscr{P}_2$ have at most 2 peaks.*
- *These (two) peaks are adjacent on the axis $A_2$.*

*Then $\mathscr{P}_1 \circledgt \mathscr{P}_2$ is single-peaked.*

**Proof.** We are going to construct an axis $A$ in a way that $\mathscr{P}_1 \circledgt \mathscr{P}_2$ is single-peaked with respect to $A$. First we split $A_2$ in two parts $A'_2$ and $A''_2$. If $\mathscr{P}_2$ contains two peaks (which have to be adjacent), we split $A_2$ in between these two peaks. If $\mathscr{P}_2$ contains only one peaks, we split $A_2$ left of the peak (this is arbitrary). The new axis $A$ is $A'_2$ followed by $A_1$ and then $A''_2$, i.e., $A'_2 > A_1 > A''_2$. The correctness proof of this construction is straight-forward. ❑

## 4.2 Relations between Notions of Nearly Single-Peakedness

Theorem 4.6 shows several inequalities that hold for the distance measures under consideration. We hereby show how these measures relate to each other. Notice that these inequalities do not have an immediate impact for a classical complexity analysis such as in Section 4.3.

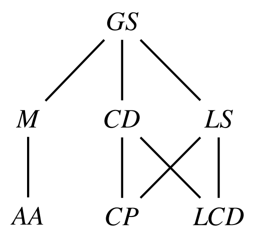
**Theorem 4.6.** *Let $\mathscr{P}$ be a preference profile. Then the following inequalities hold:*

| | | |
|---|---|---|
| *(1)* $LS(\mathscr{P}) \leq GS(\mathscr{P})$. | *(4)* $LCD(\mathscr{P}) \leq LS(\mathscr{P})$. | *(7)* $CP(\mathscr{P}) \leq CD(\mathscr{P}) + 1$. |
| *(2)* $LCD(\mathscr{P}) \leq CD(\mathscr{P})$. | *(5)* $M(\mathscr{P}) \leq GS(\mathscr{P})$. | *(8)* $CP(\mathscr{P}) \leq LS(\mathscr{P}) + 1$. |
| *(3)* $CD(\mathscr{P}) \leq GS(\mathscr{P})$. | *(6)* $AA(\mathscr{P}) \leq M(\mathscr{P})$. | |

*This list is complete in the following sense: Inequalities that are not listed here and that do not follow from transitivity do not hold in general. The resulting partial order with respect to $\leq$ is displayed in Figure 1 as a Hasse diagram.*

**Proof.** Inequalities 1 and 2 are immediate consequences from the definitions since $k$-LS allows more swaps than $k$-GS and $k$-LCD allows more candidate deletions than $k$-CD. Inequalities 3 and 4 are due to the fact that swapping two candidates in a vote is at most as effective as removing one of these candidates. Similarly, for Inequality 5 observe that removing the corresponding voter is at least as effective as swapping two candidates in the vote. Concerning Inequality 6 observe that instead of deleting a voter we can always add an additional axis for this voter. Inequality 7 follows from the fact that putting each deleted candidate in its own partition leads to single-peakedness if deleting these candidates does.

In order to show Inequality 8 let $\mathscr{P}$ be $k$-local swaps single-peaked consistent. This means that there exists an axis $A$ such that after performing at most $k$ swaps per voter, $\mathscr{P}$ becomes single-peaked with respect to $A$. Without loss of generality assume that the axis $A$ is $c_1 < c_2 < \ldots < c_n$. We now partition the candidates in $k+1$ sets $S_0, \ldots, S_k$. This is done by putting the $i$-th smallest element of $A$ into the ($i$ modulo $k+1$)-th set. Since we assume that $A$ is $c_1 < c_2 < \ldots < c_n$, we can equivalently say that $c_i$ is put into the ($i$ modulo $k+1$)-th set, i.e., the $c_1$ in $S_1$, the $c_2$ in $S_2$, the $c_k$ in $S_k$ and $c_{k+1}$ in $S_0$. Let $S \in \{S_0, \ldots, S_k\}$. Towards a contradiction assume that $\mathscr{P}[S]$ is not single-peaked with respect to $A[S]$. By Lemma 4.2 there exists some voter $v \in V$ and three candidates $c_i, c_j, c_k \in C$ such that $c_i < c_j < c_k$ on axis $A[S]$ (or equivalently $i < j < k$), $c_i \succ_v c_j$ and $c_k \succ_v c_j$. On axis $A$ the distance between $c_i$ and $c_j$ respectively $c_j$ and $c_k$ is at least $k+1$, i.e., at least $k$ elements lie in between them. We know that at most $k$ swaps in $\succ_v$ can make this profile single-peaked with respect to $A$. Let $\succ'_v$ denote this swapped vote. Necessarily these swaps have to either cause that $c_j \succ'_v c_{j-1} \succ'_v \ldots \succ'_v c_{i+1} \succ'_v c_i$ holds or that $c_j \succ'_v c_{j+1} \succ'_v \ldots \succ'_v c_{k-1} \succ'_v c_k$ holds in $\succ'_v$ (depending whether the peak of $\succ'_v$ is right or left of $c_j$). Let us focus on the case that the swaps ensure that $c_j \succ'_v c_{j-1} \succ'_v \ldots \succ'_v c_{i+1} \succ'_v c_i$ – the other case is analogous. For $\succ_v$, contrary to $\succ'_v$, it holds that $c_i \succ_v c_j$. Hence these swaps have to cause that $c_j \succ'_v c_i$ holds. In addition, at least $k$ elements, namely $c_{i+1}, \ldots, c_{j-1}$, have to be in between them. This requires at least $k+1$ swaps which contradicts the




| | | |
|---|---|---|
| GS | ... | Global Swaps |
| M | ... | Maverick |
| CD | ... | Candidate Deletion |
| LS | ... | Local Swaps |
| AA | ... | Additional Axes |
| CP | ... | Candidate Partition |
| LCD | ... | Local Candidate Deletion |

Figure 1: Hasse diagram of the partial order described in Theorem 4.6.

fact that at most $k$ swaps suffice. Therefore for all partition sets $S$, $\mathscr{P}[S]$ is single-peaked consistent and $CP(\mathscr{P}) \leq LS(\mathscr{P}) + 1$.

It remains to show that these are indeed all inequalities. This can be done by providing counterexamples for each remaining case. ❑

## 4.3 Complexity of Nearly Single-Peaked Consistency

Let us first introduce a lemma which we will use in the proofs of the theorems below.

**Lemma 4.7.** *We are given a set of candidates $C = \{a, b, c, d\}$ and three preference relations $\succ_v$, $\succ_e$ and $\succ_{ne}$, where the candidates are ranked as follows:*

- $a \succ_v c \succ_v b \succ_v d$,
- $c \succ_e b \succ_e d \succ_e a$ *and*
- $d \succ_{ne} c \succ_{ne} b \succ_{ne} a$.

*Then the preference profile $(\succ_v, \succ_e)$ is single-peaked with respect to the axis $a > c > b > d$ and $(\succ_e, \succ_{ne})$ is single-peaked with respect to the axis $d > c > b > a$. The profile $(\succ_v, \succ_{ne})$ is not single-peaked consistent.*

We start with maverick single-peaked consistency where we show NP-hardness via a reduction from the clique problem, one of the standard NP-complete problems (see, e.g., [GJ79]).

**Theorem 4.8.** MAVERICK SINGLE-PEAKED CONSISTENCY *is* NP-*complete.*

**Proof.** To show hardness we reduce from CLIQUE. Let $G = (V_G, E_G)$ be the graph in which we look for a clique of size $s$. Furthermore, let $V_G = \{v_1, \ldots, v_n\}$ be the set of vertices and $E_G$ the set of edges. Each vertex $v_i$ has four corresponding candidates $c_i^1, \ldots, c_i^4$. We consequently have $C = \{c_1^1, \ldots, c_1^4, c_2^1, \ldots, c_2^4, \ldots, c_n^1, \ldots, c_n^4\}$. The voters directly correspond to vertices. Therefore we define, by slight abuse of notation, $V = \{v_1, \ldots, v_n\}$.

In order to define the preference relations we introduce three functions creating partial votes. In the following definition let $a, b, c, d \in C$.

$$f_v(a,b,c,d) = a \succ c \succ b \succ d$$
$$f_e(a,b,c,d) = c \succ b \succ d \succ a$$
$$f_{ne}(a,b,c,d) = d \succ c \succ b \succ a$$

If we consider $f_v$, $f_e$ and $f_{ne}$ as preference relations then observe that by Lemma 4.7 $(f_v, f_e)$ and $(f_e, f_{ne})$ are single-peaked consistent but $(f_v, f_{ne})$ is not.

Next we define a mapping $p(i, j)$ to a total order over the candidates $\{c_j^1, \ldots, c_j^4\}$.

$$p(i,j) = \begin{cases} f_v(c_j^1, c_j^2, c_j^3, c_j^4) & \text{if } i = j \\ f_e(c_j^1, c_j^2, c_j^3, c_j^4) & \text{if } \{i, j\} \in E_G \\ f_{ne}(c_j^1, c_j^2, c_j^3, c_j^4) & \text{if } \{i, j\} \notin E_G \end{cases}$$

The intuition behind function $p(i, j)$ is to encode a row of the adjacency matrix of $G$ as a vote in the preference profile $\mathscr{P}$. To this end, we put in "cell" $(i, j)$ the result of $f_e$ if there is an edge between $i$ and $j$. In case there is no edge between $i$ and $j$ we put the result of $f_{ne}$ in cell $(i, j)$. In the special case $i = j$ (we are in the diagonal of the matrix) we put the result of $f_v$ in the cell.

Let the partial profiles representing the columns of the adjacency matrix be defined as $\mathscr{P}_j = (p(1,j),\dots,p(n,j))$, for $1 \leq j \leq n$. We are now going to define the preference profile $\mathscr{P} = (\succ_1,\dots,\succ_n)$ by

$$\mathscr{P} = \mathscr{P}_1 \otimes \mathscr{P}_2 \otimes \dots \otimes \mathscr{P}_n.$$

To conclude the construction let $E = (C,V,\mathscr{P})$ and $k = n - s$, i.e., we are allowed to delete $k$ mavericks from $E$ in order to obtain a single-peaked profile. The intention behind the construction is that the voters in a single-peaked profile will correspond to a clique. We claim that $G$ has a clique of cardinality $s$ if and only if it is possible to remove $k$ voters from $\mathscr{P}$ in order to make the resulting preference profile single-peaked consistent.

"$\Rightarrow$" Assume that there is a clique $I = \{v_{i_1},\dots,v_{i_s}\}$ with $|I| = s$. Let $\mathscr{P}' = (\succ_{i_1},\dots,\succ_{i_s})$. By that we keep only those voters whose corresponding vertices are contained in the clique $I$. Observe that the election $E' = (C,I,\mathscr{P}')$ can be obtained by deleting $k = n - s$ mavericks from the election $E$, $|V \setminus I| = k$. It remains to show that $E'$ is indeed single-peaked consistent. Remember that we denoted the preference relations in the $j$-th "column" of the profile by $\mathscr{P}_j$. By $\mathscr{P}'_j$ we denote the $j$-th "column" of a profile considering only the voters from $\mathscr{P}'$. Since $I$ is a clique, for each $x,y \in I$, $x \neq y$, there is an edge $\{x,y\} \in E_G$. Thus the profile cannot contain an instantiation of $f_v$ and of $f_{ne}$ in the same column. By Lemma 4.7, all profiles $\mathscr{P}_j$ with $1 \leq j \leq n$ are single-peaked consistent. In order to be able to apply Lemma 4.5, all conditions have to be checked. First, notice that the profiles $\mathscr{P}'_j$ and $\mathscr{P}'_{j'}$, for $1 \leq j < j' \leq n$, do not share any candidates and are single-peaked consistent. Furthermore, each of the profiles has at most two peaks. Each column contains either instantiations of $f_v$ and $f_e$ or instantiations of $f_e$ and $f_{ne}$. Otherwise it would not be single-peaked consistent. But then there are only two top-ranked candidates, i.e., either the candidates top-ranked by $f_v$ and $f_e$, or the candidates top-ranked by $f_e$ and $f_{ne}$. Finally, the two top-ranked candidates of $\mathscr{P}'_j$ have to be adjacent on the axis which gives single-peaked consistency. Consider again Lemma 4.7. For $(f_v, f_e)$ the top-ranked candidates $a$ and $c$ are adjacent on the axis $a > c > b > d$. The same holds for $(f_e, f_{ne})$ with axis $d > c > b > a$ and $c$, $d$ as top-ranked candidates. Since all conditions are fulfilled, we can iteratively apply Lemma 4.5. Therefore, $\mathscr{P}'_1 \otimes \mathscr{P}'_2, (\mathscr{P}'_1 \otimes \mathscr{P}'_2) \otimes \mathscr{P}'_3, \dots, (\mathscr{P}'_1 \otimes \dots) \otimes \mathscr{P}'_n$ and hence also $\mathscr{P}'$ are single-peaked consistent.

"$\Leftarrow$" Assume that $E' = (C,V',\mathscr{P}')$ is an election that has been obtained from $E$ by deleting $k$ voters such that $\mathscr{P}'$ is single-peaked. Consequently $|V'| = s$. Let $V' = \{v_{i_1},\dots,v_{i_s}\}$ and $\mathscr{P}' = (\succ_{i_1},\dots,\succ_{i_s})$.

We claim that $V'$ is a clique in $G$. By Lemma 4.3 we know that each of the $n$ columns $(\mathscr{P}'_1,\dots,\mathscr{P}'_n)$ of $\mathscr{P}'$ is single-peaked consistent. Then, by Lemma 4.7, each column must not contain an instance of $f_v$ together with an instance of $f_{ne}$. (Otherwise the respective column would not be single-peaked consistent!) Observe that by construction each vote (in $\mathscr{P}'$) contains an instance of $f_v$ in some column. But then each vertex must be adjacent to all other vertices – in other words $V'$ is a clique. ❑

We now turn to additional axes single-peaked consistency. Here we make use of a similar construction as presented in Theorem 4.8 with the difference that we now show NP-hardness via a reduction from the partition into cliques problem, which is also one of the standard NP-complete problems (see, e.g., [GJ79]).

**Theorem 4.9.** ADDITIONAL AXES SINGLE-PEAKED CONSISTENCY *is* NP*-complete.*

**Proof.** Hardness is shown by a reduction from PARTITION INTO CLIQUES. For the reduction we use the same transformation as presented in the proof of Theorem 4.8 to obtain an election. Then we set $k = s - 1$, i.e., we are searching for a partition of the voters into $s$ disjoint sets such that each of the partitions is single-peaked consistent. Due to the one-to-one correspondence between voters and vertices we can use the partition of the vertices to obtain a partition of the voters and vice versa. With arguments similar to the proof of Theorem 4.8 one can show that a set of vertices is a clique if and only if the corresponding profile is single-peaked consistent. ❑

In the proofs of our last two results, we will provide reductions from the NP-complete problem MINIMUM RADIUS, which was shown to be NP-complete in [FL97] and is defined as follows:

| MINIMUM RADIUS | |
|---|---|
| **Given:** | A set of strings $S \subseteq \{0,1\}^n$ and a positive integer $s$. |
| **Question:** | Has $S$ a radius of at most $s$, i.e., is there a string $\alpha \in \{0,1\}^n$ such that each string in $S$ has a Hamming distance of at most $s$ to $\alpha$? |

**Theorem 4.10.** LOCAL CANDIDATE DELETION SINGLE-PEAKED CONSISTENCY *is* NP-*complete.*

**Proof.** A MINIMUM RADIUS instance is given by $S \subseteq \{0,1\}^n$, the set of binary strings, and a positive integer $s$. Given a string $\beta$, let $\beta(k)$ denote the bit value at the $k$-th position in $\beta$. We are going to construct an LCD SINGLE-PEAKED CONSISTENCY instance. Each string in $S = \{\beta_1, \dots, \beta_m\}$ will correspond to a voter. Each bit of the strings corresponds to two candidates. In addition, we have $2 \cdot m \cdot s + 2$ extra candidates. Consequently, we have $C = \{c_1^1, c_1^2, c_2^1, c_2^2, \dots, c_n^1, c_n^2, c'_1, \dots, c'_{ms+1}, c''_1, \dots, c''_{ms+1}\}$.

We define the preference profile with the help of two functions creating total orders.

$$f_0(a,b) = a \succ b \qquad\qquad f_1(a,b) = b \succ a$$

The vote $\succ_k$, for each $k \in \{1, \dots, m\}$, is of the form

$$c'_1 \ \dots \ c'_{ms+1} \ \ f_{\beta_k(1)}(c_1^1, c_1^2) \ \ f_{\beta_k(2)}(c_2^1, c_2^2) \ \ \dots \ \ f_{\beta_k(n)}(c_n^1, c_n^2) \ \ c''_1 \ \dots \ c''_{ms+1}.$$

Furthermore, let $\succ_k^r$, $1 \le k \le m$, denote the reverse order of $\succ_k$. The preference profile $\mathscr{P}$ is now defined as $(\succ_1, \dots, \succ_n, \succ_1^r, \dots, \succ_n^r)$. We claim that $(V, C, \mathscr{P})$ is $s$-LCD single-peaked consistent if and only if $S$ has a radius of at most $s$.

"$\Leftarrow$" Suppose that $S$ has a radius of at most $s$, i.e., there is a string $\alpha \in \{0,1\}^n$ with Hamming distance at most $s$ to each $\beta \in S$. We consider the following axis $A$:

$$c'_1 > \dots > c'_{ms+1} > f_{\alpha(1)}(c_1^1, c_1^2) > f_{\alpha(2)}(c_2^1, c_2^2) > \dots f_{\alpha(n)}(c_n^1, c_n^2) > c''_1 > \dots > c''_{ms+1}.$$

We claim that $\mathscr{P}$ is single-peaked with respect to $A$ after deleting at most $s$ candidates in each vote. The deletions for vote $\succ_k$, $k \in \{1, \dots, m\}$, are the following: We delete candidate $c_i^1$ in $\succ_k$ if and only if $\alpha(i) \neq \beta_k(i)$. The deletions in $\succ_k^r$ are exactly the same as in $\succ_k$. These are at most $s$ deletions since the Hamming distance between $\alpha$ and every $\beta \in S$ is at most $s$. After these deletions all votes are either subsequences of $A$ or its reverse. Hence we obtain a single-peaked consistent profile.

"$\Rightarrow$" Let $\mathscr{P}'$ be the partial, single-peaked consistent profile that was obtained by deleting at most $s$ candidates in each vote. First, note that some $c' \in \{c'_1, \dots, c'_{ms+1}\}$ has not been deleted in any vote since in total at most $m \cdot s$ many different candidates can be deleted. In the same way let $c'' \in \{c''_1, \dots, c''_{ms+1}\}$ be a candidate that has not been deleted in any vote. Now let us consider the profile $\mathscr{P}'[\{c', c'', c_i^1, c_i^2\}]$ for any $i \in \{1, \dots, n\}$. We claim that $\alpha$, defined in the following way, has a Hamming distance of at most $s$ to all bitstrings in $S$.

$$\alpha(k) = \begin{cases} 0 & \text{if } \mathscr{P}' \text{ contains the vote } c' \succ c_i^1 \succ c_i^2 \succ c'', \\ 1 & \text{if } \mathscr{P}' \text{ contains the vote } c' \succ c_i^2 \succ c_i^1 \succ c'', \\ 1 & \text{otherwise.} \end{cases}$$

First, observe that case 1 and 2 cannot occur at the same time since then $\mathscr{P}'$ would not be single-peaked consistent. This is because $\mathscr{P}'[\{c', c'', c_i^1, c_i^2\}]$ also contains the vote $c'' \succ \dots \succ c'$, where the dots indicate that $c_i^1$ and $c_i^2$ might also appear in this vote (between $c''$ and $c'$). Furthermore, Let

$\beta_j \in S$ from some $j \in \{1,\dots,n\}$. Note that if at any position $i$, $\beta_j(i) \neq \alpha(i)$ then either $c_i^1$ or $c_i^2$ had to be deleted in the vote $\succ_j$. Hence the set $\{k \in \{1,\dots,m\} \mid \alpha(i) \neq \beta_j(i)\}$ cannot contain more than $s$ elements because this would require more than $s$ candidate deletions in the corresponding vote $\succ_j$. Hereby we have shown that the Hamming distance of $\alpha$ and $\beta_j$ is at most $s$. ❑

**Theorem 4.11.** LOCAL SWAPS SINGLE-PEAKED CONSISTENCY *is* NP-*complete.*

**Proof.** We use the same construction as in the proof of Theorem 4.10. It holds that $(V, C, \mathscr{P})$ is $s$-LS single-peaked consistent if and only if $S$ has a radius of at most $s$. This can be shown similarly to the proof of Theorem 4.10 except that we swap elements instead of deleting them. ❑

# 5 Conclusions and Open Questions

We have investigated the problem of nearly single-peaked consistency. To this end, we have formally defined two notions of nearly single-peakedness suggested by Escoffier, Lang, and Öztürk [ELÖ08]. Furthermore, we have introduced three new notions of nearly single-peakedness. We have drawn a complete picture of the relations between all the notions of nearly single-peakedness discussed in this paper. For four notions we have shown that deciding single-peaked consistency is NP-complete. An obvious direction for future work is to pinpoint the complexity of the remaining three problems. It is noteworthy in this regard that a distance measure has been studied very recently which admits a polynomial time algorithm for nearly single-peaked consistency [EFS12].

NP-completeness, however, does not rule out the possibility of algorithms that perform well in practice. One approach is to search for fixed-parameter algorithms. For example, it might be that $k$-maverick single-peaked consistency can be decided by a fixed-parameter algorithm, i.e., an algorithm with runtime $f(k) \cdot poly(n)$ for some computable function $f$. A second approach is the development of approximation algorithms since nearly single-peaked consistency can also be seen as an optimization problem.

Another interesting direction for future work is extending our models to manipulative behavior, such as manipulation, control, and bribery. That is, assuming we have a nearly single-peaked electorate according to one of our notions, how hard is a manipulative action under a certain voting rule computationally? This line of work has already been started in [FHH11] for some distance measures. Finally, there might be further useful and natural distance measures regarding single-peakedness to be found.

**Acknowledgments:** We thank the anonymous reviewers for their helpful comments.

# References

[BBHH10] F. Brandt, M. Brill, E. Hemaspaandra, and L. A. Hemaspaandra. Bypassing combinatorial protections: Polynomial-time algorithms for single-peaked electorates. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI-10)*, pages 715–722. AAAI Press, 2010.

[BEH+10] D. Baumeister, G. Erdélyi, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Computational aspects of approval voting. In J. Laslier and R. Sanver, editors, *Handbook on Approval Voting*, chapter 10, pages 199–251. Springer, 2010.

[Bla48] D. Black. On the rationale of group decision making. *Journal of Political Economy*, 56(1):23–34, 1948.

[BT86] J. Bartholdi and M. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165 – 169, 1986.

[BTT89] J. Bartholdi, C. Tovey, and M. Trick. The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6(3):227–241, 1989.

[BTT92] J. Bartholdi, C. Tovey, and M. Trick. How hard is it to control an election? *Mathematical and Computer Modeling*, 16(8/9):27–40, 1992.

[Con09] V. Conitzer. Eliciting single-peaked preferences using comparison queries. *Journal of Artificial Intelligence Research*, 35:161–191, 2009.

[DKNS01] C. Dwork, R. Kumar, M. Naor, and D. Sivakumar. Rank aggregation methods for the web. In *Proceedings of the 10th International World Wide Web Conference*, pages 613–622. ACM Press, 2001.

[EFS09] E. Elkind, P. Faliszewski, and A. M. Slinko. Swap bribery. In *Proceedings of the 2nd International Symposium on Algorithmic Game Theory (SAGT-2009)*, pages 299–310, 2009.

[EFS12] E. Elkind, P. Faliszewski, and A. M. Slinko. Clone structures in voters' preferences. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC-12)*, pages 496–513. ACM, 2012.

[ELÖ08] B. Escoffier, J. Lang, and M. Öztürk. Single-peaked consistency and its complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence (ECAI 2008)*, volume 178 of *Frontiers in Artificial Intelligence and Applications*, pages 366–370. IOS Press, 2008.

[ER97] E. Ephrati and J. Rosenschein. A heuristic technique for multi-agent planning. *Annals of Mathematics and Artificial Intelligence*, 20(1–4):13–67, 1997.

[FHH09] P. Faliszewski, E. Hemaspaandra, and L. A. Hemaspaandra. How hard is bribery in elections? *Journal of Artificial Intelligence Research*, 35:485–532, 2009.

[FHH11] P. Faliszewski, E. Hemaspaandra, and L. A. Hemaspaandra. The complexity of manipulative attacks in nearly single-peaked electorates. In *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge (TARK-2011)*, pages 228–237, 2011.

[FHHR11] P. Faliszewski, E. Hemaspaandra, L. A. Hemaspaandra, and J. Rothe. The shield that never was: Societies with single-peaked preferences are more open to manipulation and control. *Information and Computation*, 209(2):89–107, 2011.

[FL97] M. Frances and A. Litman. On covering problems of codes. *Theory of Computing Systems*, 30:113–119, 1997.

[GJ79] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979.

[GMHS99] S. Ghosh, M. Mundhe, K. Hernandez, and S. Sen. Voting for movies: The anatomy of recommender systems. In *Proceedings of the 3rd Annual Conference on Autonomous Agents*, pages 434–435. ACM Press, 1999.

[MN08] T. Meskanen and H. Nurmi. Closeness counts in social choice. In M. Braham and F. Steffen, editors, *Power, Freedom, and Voting*, pages 289–306. Springer, 2008.

[Wal07] T. Walsh. Uncertainty in preference elicitation and aggregation. In *Proceedings of the 22nd AAAI Conference on Artificial Intelligence (AAAI-07)*, pages 3–8. AAAI Press, 2007.



Gábor Erdélyi
Institut für Betriebswirtschaftslehre
Universität Siegen
Siegen, Germany
Email: `erdelyi@wiwi.uni-siegen.de`

Martin Lackner
Vienna University of Technology
Vienna, Austria
Email: `lackner@dbai.tuwien.ac.at`

Andreas Pfandler
Vienna University of Technology
Vienna, Austria
Email: `pfandler@dbai.tuwien.ac.at`